Finite Volume Method with Coordinate Transformation between Two Curvilinear Coordinate Systems

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1 Introduction

In this article, I will discuss the detailed implementation of the finite volume method (FVM) with coordinate transformation between two curvilinear coordinate systems.

The reason that why I care about coordinate transformation between curvilinear coordinate systems is that, in my Ph.D. thesis I have to transform an axial asymmetric geometry from a cylindrical coordinate system to a new curvilinear coordinate system so that the transformed geometry is axial symmetric.

To be honest, it really took me great effort to accomplish this techniques. It turns out that, back to the days when I spent all my time trying and struggling to figure out what is the right expressions of the Gaussian integral in different coordinate systems, I really lacked the experiences and mathematical skills that related to curvilinear coordinate systems. In fact, the content of my graduate education did not cover this topic at all. Therefore, I had to walk through the mine field by myself, otherwise I won't have the chance to finish my Ph.D. thesis. Actually, I was not alone through all that period of time. I have to thank my colleague, **Bing DONG <dongbing@sjtu.edu.cn>**, who spent his valuable time with me, again and again, to derive the mathematical expressions on the white board in our lab. We together tried really hard to understand what a curvilinear coordinate system really is. We exchanged thoughts and conducted mathematical experiments, revealing the mysterious characteristics of curvilinear coordinate system, bit by bit. And we both found that it was worth the time and effort. I was really happy that you were with me back to those days, buddy.

2 Vector and Coordinate Transformation

A coordinate transformation affects the way how vectors are represented as coordinates, and the way we perform vector calculation. Since vector calculation is essential for theory of fluid dynamics and computational fluid dynamics (CFD), it would be better that we discuss this topic right at the beginning.

In this section, the relation between coordinate transformation and vector calculations is discussed. First a demonstration problem is introduced in which the coordinate transformation between curvilinear coordinate systems is needed. In fact, it is the actual work of my Ph.D. thesis.

2.1 Coordinate transformation and partial derivatives

As illustrated in Fig. 1, the original geometry is a 2D annular consists of two circles which are depicted by solid lines. I need a coordinate transformation to turn this axial asymmetric geometry region into an axial symmetric one, similar to that represent by the dashed line and outer circle. In the original work, the outer circle and inner circle are defined as the stator and rotor, respectively.

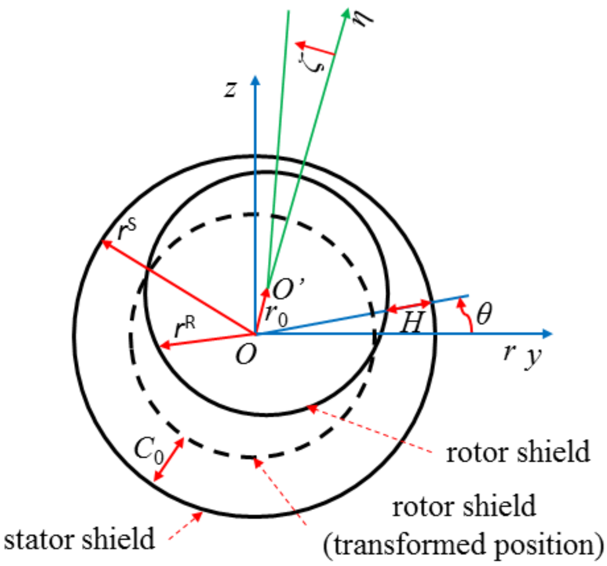

Fig. 1 Coordinate transformation

The original problem is described in cylindrical coordinate system, (x, r, θ), with the origin, O, at the center of the stator and the x-axis be along the axis of the stator. (x, r, θ) is transformed to a new coordinate system (ξ, η, ζ) by the expressions of Eq. (1.1) to (1.3).

$$\xi = x \tag{1.1}$$

$$\eta = r^S - \frac{r^S - r}{H} C_0 \tag{1.2}$$

$$\zeta = \theta \tag{1.3}$$

where Eq. (1.2) is the same with that used by Dietzen and Nordmann[1, 2]. The coordinate transformation of Eq. (1.1) to (1.3) is only valid when the ratio of r_0/C_0 is relatively small, say about 0.1. Fig. 2 shows an example of this coordinate transformation. In Fig. 2, the red lines are the original stator and rotor profile. The green lines represent a sample mesh in the original coordinate system. The blue dashed line is the transformed rotor profile. After the coordinate transformation, the green lines will be placed in an axial symmetric manner.

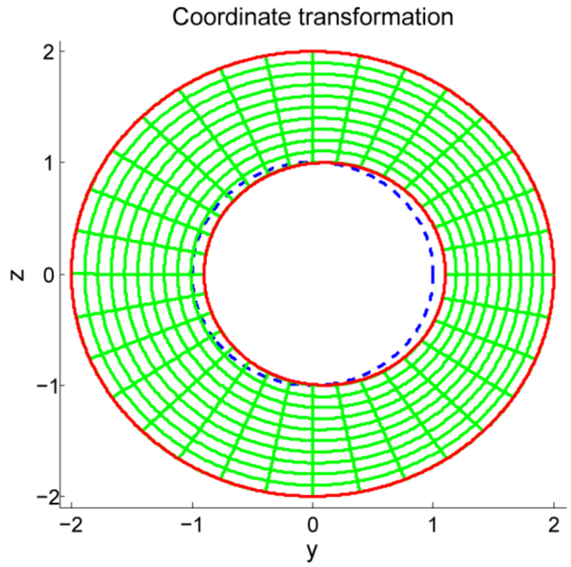

Fig. 2 Illustration of coordinate transformation.

In Eq. (1.2), H is written as

$$H = C_0 + \varepsilon_e h_1 \tag{1.4}$$

where C_0 and ε_e are geometry constants. h_1 is a function of ζ. Based on Eq. (1.4) and (1.2) we can obtain Eq. (1.5).

$$r = \eta - \varepsilon_e \left(r^S - \eta \right) \frac{h_1}{C_0} \tag{1.5}$$

As for the partial derivatives in the N-S equations, Eq. (1.6) and (1.7) holds.

$$\begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial r} \\ \dfrac{\partial}{\partial \theta} \end{Bmatrix} = \left[\mathbf{J}^{-1} \right] \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \\ \dfrac{\partial}{\partial \zeta} \end{Bmatrix} \tag{1.6}$$

$$\left(\frac{\partial \phi}{\partial t} \right)_r = \left(\frac{\partial \phi}{\partial t} \right)_\eta + \frac{\partial \phi}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial \phi}{\partial \eta}\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial \zeta}\frac{\partial \zeta}{\partial t} \tag{1.7}$$

where [**J**] is the Jacobian matrix.

$$[\mathbf{J}] = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial r}{\partial \xi} & \dfrac{\partial \theta}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial r}{\partial \eta} & \dfrac{\partial \theta}{\partial \eta} \\ \dfrac{\partial x}{\partial \zeta} & \dfrac{\partial r}{\partial \zeta} & \dfrac{\partial \theta}{\partial \zeta} \end{bmatrix} \tag{1.8}$$

From the chain rule, [$\mathbf{J}^{-1}$] could be expressed as Eq. (1.9).

$$[\mathbf{J}^{-1}] = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} & \dfrac{\partial \zeta}{\partial x} \\ \dfrac{\partial \xi}{\partial r} & \dfrac{\partial \eta}{\partial r} & \dfrac{\partial \zeta}{\partial r} \\ \dfrac{\partial \xi}{\partial \theta} & \dfrac{\partial \eta}{\partial \theta} & \dfrac{\partial \zeta}{\partial \theta} \end{bmatrix} \tag{1.9}$$

Values of some of the terms in Eq. (1.6) and (1.7) are relatively easy to be determined, and are expressed by Eq. (1.10) and (1.11).

$$[\mathbf{J}] = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial r}{\partial \xi} & \dfrac{\partial \theta}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial r}{\partial \eta} & \dfrac{\partial \theta}{\partial \eta} \\ \dfrac{\partial x}{\partial \zeta} & \dfrac{\partial r}{\partial \zeta} & \dfrac{\partial \theta}{\partial \zeta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{\partial r}{\partial \eta} & 0 \\ 0 & \dfrac{\partial r}{\partial \zeta} & 1 \end{bmatrix} \tag{1.10}$$

$$\frac{\partial \xi}{\partial t} = \frac{\partial \zeta}{\partial t} = 0 \tag{1.11}$$

From now on, the short-handed version of $\partial r / \partial \eta$ and $\partial r / \partial \zeta$ are used. The similar notations could be found in [3].

$$r_{,\eta} = \frac{\partial r}{\partial \eta} = 1 + \varepsilon_{\mathrm{e}} \frac{h_1}{C_0} \tag{1.12}$$

$$r_{,\zeta} = \frac{\partial r}{\partial \zeta} = \frac{\partial}{\partial \zeta}\left(\eta - \varepsilon_{\mathrm{e}} \left(r^{\mathrm{S}} - \eta \right) \frac{h_1}{C_0} \right) = -\frac{\varepsilon_{\mathrm{e}}}{C_0} \left(r^{\mathrm{S}} - \eta \right) \frac{\partial h_1}{\partial \zeta} \tag{1.13}$$

Then the determinant of [$\mathbf{J}$] is

$$J = \det\left(\mathbf{J}\right) = r_{,\eta} = 1 + \varepsilon_{\mathrm{e}} \frac{h_1}{C_0} \tag{1.14}$$

Furthermore

$$\left[\mathbf{J}^{-1}\right] = \frac{1}{J}\left[\mathbf{J}^{*}\right] = \underbrace{\frac{1}{J}\begin{bmatrix} r_{,\eta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -r_{,\zeta} & r_{,\eta} \end{bmatrix}}_{*} = \frac{1}{J}\begin{bmatrix} J & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -r_{,\zeta} & J \end{bmatrix} \tag{1.15}$$

At this point, if the terms marked by * in Eq. (1.15) are used, then the partial derivatives of an arbitrary scalar ϕ variable could be expressed as

$$\frac{\partial \phi}{\partial x} = \frac{1}{J} r_{,\eta} \frac{\partial \phi}{\partial \xi} \tag{1.16}$$

$$\frac{\partial \phi}{\partial r} = \frac{1}{J} \frac{\partial \phi}{\partial \eta} \tag{1.17}$$

$$\frac{\partial \phi}{\partial \theta} = \frac{1}{J}\left(-r_{,\zeta} \frac{\partial \phi}{\partial \eta} + r_{,\eta} \frac{\partial \phi}{\partial \zeta} \right) \tag{1.18}$$

$$\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial \eta} \frac{\partial \eta}{\partial t} \tag{1.19}$$

And we must not forget the important infinitesimal term of Eq. (1.20).

$$r\mathrm{d}\theta \mathrm{d}r \mathrm{d}x = r\left(\eta, \zeta\right) J \mathrm{d}\zeta \mathrm{d}\eta \mathrm{d}\xi \tag{1.20}$$

2.2 The unit base vectors in cylindrical coordinate system

A vector is a vector, it dose not depend on the way it is described. However, in general 3D space, people agree on that three particular vectors in the Cartesian coordinate system are chosen to be base vectors, Eq. (1.21). Once the base vectors are defined any vector of the same space could be expressed by a linear combination of those base vectors. It is a fundamental result of the theory of linear algebra. It is notable that Eq. (1.21) is a set of unit vectors that orthogonal with each other.

$$\mathbf{e}_{\mathrm{x}}=\begin{Bmatrix}1\\0\\0\end{Bmatrix}=\mathbf{e}_1,\mathbf{e}_{\mathrm{y}}=\begin{Bmatrix}0\\1\\0\end{Bmatrix}=\mathbf{e}_2,\mathbf{e}_{\mathrm{z}}=\begin{Bmatrix}0\\0\\1\end{Bmatrix}=\mathbf{e}_3 \tag{1.21}$$

For 3D cylindrical coordinate system (x, r, θ), the unit base vector could be defined as Eq. (1.22).

$$\begin{aligned}\mathbf{e}_{\mathrm{x}}&=\begin{Bmatrix}1\\0\\0\end{Bmatrix}\\ \mathbf{e}_{\mathrm{r}}&=\begin{Bmatrix}0\\ \cos(\theta)\\ \sin(\theta)\end{Bmatrix}=\cos(\theta)\mathbf{e}_{\mathrm{y}}+\sin(\theta)\mathbf{e}_{\mathrm{z}}\\ \mathbf{e}_{\theta}&=\begin{Bmatrix}0\\ -\sin(\theta)\\ \cos(\theta)\end{Bmatrix}=-\sin(\theta)\mathbf{e}_{\mathrm{y}}+\cos(\theta)\mathbf{e}_{\mathrm{z}}\end{aligned} \tag{1.22}$$

where the three numbers in each curly brace are the coordinates with respect to the Cartesian coordinate system, (x,y,z). It is worth to be noted that Eq. (1.22) shows that the unit vectors are functions of θ.

Similar to the case in the Cartesian coordinate, a general vector **v** could be expressed by the linear combination of the base vectors, as shown by Eq. (1.23).

$$\mathbf{v}=v_{\mathrm{x}}\mathbf{e}_{\mathrm{x}}+v_{\mathrm{r}}\mathbf{e}_{\mathrm{r}}+v_{\theta}\mathbf{e}_{\theta} \tag{1.23}$$

Then the addition and inner product (or dot product) operations between vectors in 3D cylindrical coordinate could be defined in a straight forward way.

Considering Eq. (1.23), two special aspects should be noted.

（1） Before any operation of addition or inner product is taken between vectors, it should be verified that the vectors are referring to their particular set of base vectors. This constraint on the operations of addition and inner product is the direct consequence of the fact that the base vectors used in Eq. (1.23) are functions of special position. A situation with particular interest is that, two vectors, which participate an operation of addition of inner product, share the same set of base vectors. Then the expressions of Eq. (1.24) and (1.25) hold.

$$\begin{aligned}\mathbf{v}_1+\mathbf{v}_2&=v_{\mathrm{x1}}\mathbf{e}_{\mathrm{x}}+v_{\mathrm{r1}}\mathbf{e}_{\mathrm{r}}+v_{\theta1}\mathbf{e}_{\theta}+v_{\mathrm{x2}}\mathbf{e}_{\mathrm{x}}+v_{\mathrm{r2}}\mathbf{e}_{\mathrm{r}}+v_{\theta2}\mathbf{e}_{\theta}\\ &=\left(v_{\mathrm{x1}}+v_{\mathrm{x2}}\right)\mathbf{e}_{\mathrm{x}}+\left(v_{\mathrm{r1}}+v_{\mathrm{r2}}\right)\mathbf{e}_{\mathrm{r}}+\left(v_{\theta1}+v_{\theta2}\right)\mathbf{e}_{\theta}\end{aligned} \tag{1.24}$$

$$\begin{aligned}\mathbf{v}_1 \cdot \mathbf{v}_2 &= \left(v_{x1}\mathbf{e}_x + v_{r1}\mathbf{e}_r + v_{\theta1}\mathbf{e}_\theta\right)\cdot\left(v_{x2}\mathbf{e}_x + v_{r2}\mathbf{e}_r + v_{\theta2}\mathbf{e}_\theta\right) \\ &= v_{x1}v_{x2} + v_{r1}v_{r2} + v_{\theta1}v_{\theta2}\end{aligned} \tag{1.25}$$

with the fact that

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \text{ with } i, j = \text{x, r, } \theta \tag{1.26}$$

Thus the three vectors which belong to the same set of base vectors in a cylindrical coordinate system preserve the orthonormal property.

For convenience, unless stated otherwise, in the following sections the operations of addition and inner product between vectors are the same with Eq. (1.24) and (1.25). Most of the time, the base vectors encountered in this article are defined by referring to the center of the control volume (CV) or the center of a face of the CV.

（2）Unlike the Cartesian coordinate system, the base vectors of Eq. (1.23) do not have the same “units” of “dimensions” if you want to force such a concept to be associated with the base vectors. It is easy to find that $\mathbf{e}_x$ and $\mathbf{e}_r$ have a sense of “linear length”, however, $\mathbf{e}_\theta$ represents a direction of angle increase anticlockwise. $\mathbf{e}_\theta$ does not have the sense of “linear length”. To make it has the sense of linear length, $\mathbf{e}_\theta$ should be associated with a parameter representing the idea of “radius”. In calculus we use $r\mathbf{e}_\theta$. In fact, $r\mathbf{e}_\theta$ represents the unit or direction of an increasing curve, not a straight line. I think the above concepts are of great importance for mathematical derivations in cylindrical coordinate system. It becomes easier when we are trying to interpret and understand calculus concepts, such as differentiation and integration, in cylindrical coordinate system if the combination of $r\mathbf{e}_\theta$ is used to think of “length”.

2.3 Transformation of vector

Before we discuss the numerical implementation of FVM, we still need to figure out how a general vector behaves during a coordinate transformation. The content discussed in this section is similar with those one could find in a book of theory of FVM with topics associated with curvilinear coordinates or body-fitted coordinates. I have to admit that those topics are seemed to be obsolete in modern times, since we are so familiar and used to FVM techniques with unstructured grids. However, for me and the specific purpose code that I would like to develop for my Ph.D. thesis, the theories of coordinate transformation and the behavior of a general vector are important.

As mentioned before, a vector is a vector, it does not change according to the coordinate system, what changes is the expression of the vector. In order to represent a vector $\mathbf{v}$ in the transformed coordinate system, (ξ, η, ζ), and at the same time, preserving the invariance of $\mathbf{v}$, theories of contravariant and covariant bases are needed. These topics are usually covered in tensor analysis or differential geometry.

Take a point in the original cylindrical coordinate system with the coordinates (x,r,θ), we can write its position vector as Eq. (1.27).

$$\mathbf{R} = x\mathbf{e}_\mathrm{x} + r\mathbf{e}_\mathrm{r} \tag{1.27}$$

It is notable that the circumferential component of **R** is always zero. According to the work of Hung[4], for the transformation between coordinate system (x, r, θ) and (ξ, η, ζ), the covariant base vectors could be defined as

$$\mathbf{g}_1 = \frac{\partial \mathbf{R}}{\partial \xi} = \frac{\partial}{\partial \xi}\left(x\mathbf{e}_\mathrm{x}\right) + \frac{\partial}{\partial \xi}\left(r\mathbf{e}_\mathrm{r}\right) = \mathbf{e}_\mathrm{x} \tag{1.28}$$

$$\mathbf{g}_2 = \frac{\partial \mathbf{R}}{\partial \eta} = \frac{\partial}{\partial \eta}\left(x\mathbf{e}_\mathrm{x}\right) + \frac{\partial}{\partial \eta}\left(r\mathbf{e}_\mathrm{r}\right) = r_{,\eta}\mathbf{e}_\mathrm{r} \tag{1.29}$$

$$\mathbf{g}_3 = \frac{\partial \mathbf{R}}{\partial \zeta} = \frac{\partial}{\partial \zeta}\left(x\mathbf{e}_\mathrm{x}\right) + \frac{\partial}{\partial \zeta}\left(r\mathbf{e}_\mathrm{r}\right) = r_{,\zeta}\mathbf{e}_\mathrm{r} + r\mathbf{e}_\theta \tag{1.30}$$

And the contravariant base vectors are

$$\mathbf{g}^1 = \nabla \xi = \frac{\partial \xi}{\partial x}\mathbf{e}_\mathrm{x} + \frac{\partial \xi}{\partial r}\mathbf{e}_\mathrm{r} + \frac{1}{r}\frac{\partial \xi}{\partial \theta}\mathbf{e}_\theta = \frac{1}{J} r_{,\eta}\mathbf{e}_\mathrm{x} \tag{1.31}$$

$$\mathbf{g}^2 = \nabla \eta = \frac{\partial \eta}{\partial x}\mathbf{e}_\mathrm{x} + \frac{\partial \eta}{\partial r}\mathbf{e}_\mathrm{r} + \frac{1}{r}\frac{\partial \eta}{\partial \theta}\mathbf{e}_\theta = \frac{1}{J}\mathbf{e}_\mathrm{r} + \frac{1}{Jr}\left(-r_{,\zeta}\right)\mathbf{e}_\theta \tag{1.32}$$

$$\mathbf{g}^3 = \nabla \zeta = \frac{\partial \zeta}{\partial x}\mathbf{e}_\mathrm{x} + \frac{\partial \zeta}{\partial r}\mathbf{e}_\mathrm{r} + \frac{1}{r}\frac{\partial \zeta}{\partial \theta}\mathbf{e}_\theta = \frac{1}{Jr} r_{,\eta}\mathbf{e}_\theta \tag{1.33}$$

where $r_{,\eta}/J = 1$, but I prefer to keep the form of Eq. (1.31) and (1.33). The covariant and contravariant base vectors satisfy the orthonormal property, Eq. (1.34).

$$\mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \text{ with } i, j = 1, 2, 3 \tag{1.34}$$

where $\delta^i{}_j$ is the Kronecker delta. However, it should be also noted that the orthonormal property does not hold among the covariant base vectors or the contravariant base vectors. Thus, coordinate system (ξ, η, ζ) is a curvilinear coordinate system with non-orthogonal bases. As long as being bases vectors, a general vector **v** can be expressed by their linear combinations.

$$\mathbf{v} = v^i \mathbf{g}_i \tag{1.35}$$

$$\mathbf{v} = v_i \mathbf{g}^i \tag{1.36}$$

where the repetition of the sub or super script i denotes the "summation" operation commonly used in tensor analysis. Here v^i is the contravariant component and v_i is the covariant component of the vector **v**. They are listed in Table 1.

$$v^i = \mathbf{v} \cdot \mathbf{g}^i \tag{1.37}$$

$$v_i = \mathbf{v} \cdot \mathbf{g}_i \tag{1.38}$$

Table 1 Contravariant & covariant components of vector **v**.

i	v^i	v_i
1	$\frac{1}{J} r_{,\eta} v_{\mathrm{x}}$	v_{x}
2	$\frac{1}{J} v_{\mathrm{r}} + \frac{1}{Jr}\left(-r_{,\zeta}\right) v_{\theta}$	$r_{,\eta} v_{\mathrm{r}}$
3	$\frac{1}{Jr} r_{,\eta} v_{\theta}$	$r_{,\zeta} v_{\mathrm{r}} + r v_{\theta}$

A transformation tensor, **Q**, could be defined based on the covariant base vectors, Eq. (1.28) to (1.30). It is easy to verify that **Q** represents a tensor which transforms Eq. (1.21) into Eq. (1.28) to (1.30), and is expressed by Eq. (1.39).

$$\mathbf{g}_i = \mathbf{Q} \bullet \mathbf{e}_i, \quad i = 1, 2, 3 \tag{1.39}$$

The matrix form of tensor Q is Eq. (1.40).

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r_{,\eta} \cos(\theta) & r_{,\zeta} \cos(\theta) - r \sin(\theta) \\ 0 & r_{,\eta} \sin(\theta) & r_{,\zeta} \sin(\theta) + r \cos(\theta) \end{bmatrix} \tag{1.40}$$

Let the determinant of **Q** be J^{tr}.

$$J^{\mathrm{tr}} = r_{,\eta} r = Jr \tag{1.41}$$

From the geometric interpretation of the covariant base vectors, they are closely related to J^{tr}, e.g. Eq. (1.42).

$$\mathbf{g}_3 \cdot \left(\mathbf{g}_1 \times \mathbf{g}_2 \right) = J^{\mathrm{tr}} \tag{1.42}$$

where "×" represents the cross product of two vectors. Recall the orthogonality between $\mathbf{g}^i$ and $\mathbf{g}_i$, Eq. (1.34), we could know that $\mathbf{g}^3$ is in the same direction of $\mathbf{g}_1 \times \mathbf{g}_2$. Therefore, a scalar parameter $\alpha^{\mathbf{g}}$ should exist so that

$$\mathbf{g}^3 = \alpha^{\mathbf{g}} \left(\mathbf{g}_1 \times \mathbf{g}_2 \right) \tag{1.43}$$

Considering Eq. (1.42), we can find

$$\mathbf{g}_3 \cdot \mathbf{g}^3 = \alpha^{\mathbf{g}} \mathbf{g}_3 \cdot \left(\mathbf{g}_1 \times \mathbf{g}_2 \right) = \alpha^{\mathbf{g}} J^{\mathrm{tr}} = 1 \tag{1.44}$$

Obviously $\alpha^{\mathbf{g}} = 1/J_{\mathrm{tr}}$. Then we have

$$J^{\mathrm{tr}} \mathbf{g}^3 = \mathbf{g}_1 \times \mathbf{g}_2 \tag{1.45}$$

It is easy to verify that

$$J^{\mathrm{tr}} \mathbf{g}^1 = \mathbf{g}_2 \times \mathbf{g}_3 \tag{1.46}$$

$$J^{\mathrm{tr}} \mathbf{g}^2 = \mathbf{g}_3 \times \mathbf{g}_1 \tag{1.47}$$

From a geometrical point of view, and based on the work of Kwak and Kiris [5], we could see that in 3D space, the module of vector $\mathbf{g}_i \times \mathbf{g}_j$ equals the area of the unit surface element whose normal vector is also $\mathbf{g}_i \times \mathbf{g}_j$. And the infinitesimal surface elements with their normal vector pointing to the ξ, η and ζ directions are

$$\begin{cases} (\mathbf{n}\mathrm{d}s)^{\xi} = \left(\dfrac{\partial \mathbf{R}}{\partial \eta} \mathrm{d}\eta \right) \times \left(\dfrac{\partial \mathbf{R}}{\partial \zeta} \mathrm{d}\zeta \right) = \mathbf{g}_2 \times \mathbf{g}_3 \mathrm{d}\eta \mathrm{d}\zeta = J^{\mathrm{tr}} \mathbf{g}^1 \mathrm{d}\eta \mathrm{d}\zeta \\ (\mathbf{n}\mathrm{d}s)^{\eta} = \left(\dfrac{\partial \mathbf{R}}{\partial \zeta} \mathrm{d}\zeta \right) \times \left(\dfrac{\partial \mathbf{R}}{\partial \xi} \mathrm{d}\xi \right) = \mathbf{g}_3 \times \mathbf{g}_1 \mathrm{d}\zeta \mathrm{d}\xi = J^{\mathrm{tr}} \mathbf{g}^2 \mathrm{d}\zeta \mathrm{d}\xi \\ (\mathbf{n}\mathrm{d}s)^{\zeta} = \left(\dfrac{\partial \mathbf{R}}{\partial \xi} \mathrm{d}\xi \right) \times \left(\dfrac{\partial \mathbf{R}}{\partial \eta} \mathrm{d}\eta \right) = \mathbf{g}_1 \times \mathbf{g}_2 \mathrm{d}\xi \mathrm{d}\eta = J^{\mathrm{tr}} \mathbf{g}^3 \mathrm{d}\xi \mathrm{d}\eta \end{cases} \tag{1.48}$$

where, again, $\mathbf{n}$ is the unit normal vector of the surface element. Eq. (1.48) has significant importance when we derive the integral over a CV in coordinate system (ξ, η, ζ).

Until this point we have discussed the important building blocks for derivation of FVM with coordinate transformation between curvilinear coordinate systems. I want to emphasis that the above content is not the same with that discussed in classical books on FVM. In classical FVM theory, the coordinate transformation is almost always performed between Cartesian coordinate system and curvilinear counterpart. In those derivations things are simple and straight forward, and there is no need to start from the basics of vectors. But the obtained results are too concise for one to grasp the fundamental essence of coordinate transformation. Conversely, for coordinate transformation between curvilinear coordinate systems, we should always start from the very basics of vectors, their expressions and calculations.

Now we have all we need, let’s move to the topic of FVM.

3 The grid

The fluid domain should be discretized into cells and grid to form control volumes. The specific fluid domain used in the current article possesses an annular shape, as illustrated in Fig. 1. A grid with fully orthogonal hexahedral cells is utilized. As for “fully orthogonal” I mean that every edge of any single cell is parallel to the base vectors of the coordinate system (ξ, η, ζ). A sample hexahedral cell is shown in Fig. 3. From now on, we could use the term “control volume” to refer to this cell. Following the convention described in the work of Versteeg and Malalasekera[6], define the 6 directions to be denoted by E(east), W(west), N(north), S(south), T(top) and B(bottom). Further, E to B are also used to indicate the 6 neighbouring points of a center point P. WE, SN and BT directions are used as the ξ, η and ζ directions. For each face of the CV, e, w, n, s, t and b are used as the face indices.

Although the cell is hexahedron, the grid points are not evenly distributed. The primary reason is that the grid points has to be clustered near the rotor and stator surfaces to fulfill the requirements of the turbulence model and to resolve the steep gradients of the physical properties in the near-wall regions. In the development of the specific code, some techniques of FVM with unstructured grid are adopted to cope with the situation of unevenly distributed grid points.

An working example 2D grid is shown in Fig. 4.

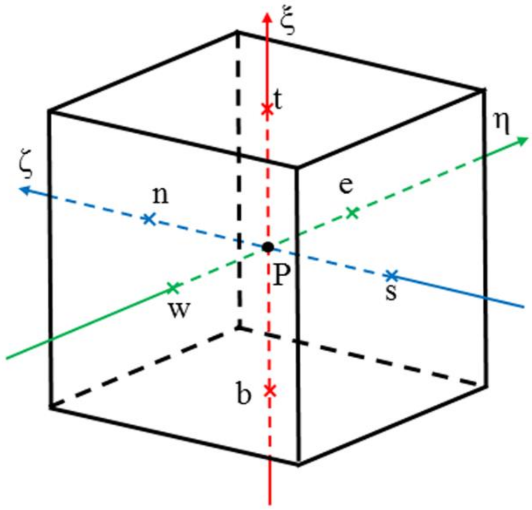

Fig. 3 Hexahedral grid cell. Control volume

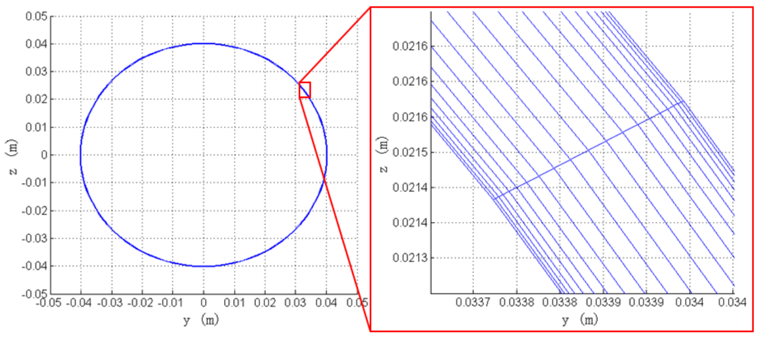

Fig. 4 Working example of grid generation for annular clearance geometry.

A storage strategy of co-located grid is used, meaning that all the physical properties including velocity, pressure and turbulence properties are stored at the center point of a cell.

4 Governing equations

The simulation material is isothermal incompressible fluid. The governing equations consist of three momentum equations and one continuity equation.

$$\rho \nabla \cdot \mathbf{u} = 0 \tag{1.49}$$

$$\rho \left[\frac{\partial \phi}{\partial t} + \nabla \cdot (\mathbf{u}\phi) \right] = \nabla \cdot (\mu_{\mathrm{eff}} \nabla \phi) + s_{\phi}^{\mathrm{m}} \tag{1.50}$$

where ϕ is an arbitrary scalar, and $\mathbf{u}$ is the velocity vector $[u, v, w]^{\mathrm{T}}$. μ_{eff} is the effective viscosity. It is the sum of the dynamic viscosity μ and the turbulent viscosity μ_{t}.

$$\mu_{\mathbf{eff}} = \mu + \mu_{\mathbf{t}} \tag{1.51}$$

The linear operators in Eq. (1.49) and (1.50) are as follows. They are all expressed in cylindrical coordinate system.

$$\nabla = \frac{\partial}{\partial x}\mathbf{e}_{\mathrm{x}} + \frac{\partial}{\partial r}\mathbf{e}_{\mathrm{r}} + \frac{1}{r}\frac{\partial}{\partial \theta}\mathbf{e}_{\theta} \tag{1.52}$$

$$\nabla \cdot \mathbf{v} = \frac{\partial v_{\mathrm{x}}}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}\left(r v_{\mathrm{r}}\right) + \frac{1}{r}\frac{\partial v_{\theta}}{\partial \theta} \tag{1.53}$$

$$\nabla \cdot \left(\Gamma \nabla\right) = \frac{\partial}{\partial x}\left(\Gamma \frac{\partial}{\partial x}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\Gamma r \frac{\partial}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\Gamma \frac{1}{r}\frac{\partial}{\partial \theta}\right) \tag{1.54}$$

where **v** is a general vector and Γ represents a diffusion parameter. The other symbols in Eq. (1.50) are listed in Table 2.

Table 2 Momentum equations.

ϕ	s_{ϕ}^{m}
u	$s_u^{\mathrm{m}} = -\frac{\partial p_{\mathrm{eff}}}{\partial x} + \frac{\partial}{\partial x}\left(\mu_{\mathrm{eff}}\frac{\partial u}{\partial x}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\mu_{\mathrm{eff}}\frac{\partial v}{\partial x}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\mu_{\mathrm{eff}}\frac{\partial w}{\partial x}\right)$
v	$s_v^{\mathrm{m}} = -\frac{\partial p_{\mathrm{eff}}}{\partial r} - \mu_{\mathrm{eff}}\frac{2}{r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{v}{r}\right) + \frac{\partial}{\partial x}\left(\mu_{\mathrm{eff}}\frac{\partial u}{\partial r}\right)$ $+\frac{1}{r}\frac{\partial}{\partial r}\left(r\mu_{\mathrm{eff}}\frac{\partial v}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(r\mu_{\mathrm{eff}}\frac{\partial}{\partial r}\left(\frac{w}{r}\right)\right) + \rho\frac{ww}{r}$
w	$s_w^{\mathrm{m}} = -\frac{1}{r}\frac{\partial p_{\mathrm{eff}}}{\partial \theta} + \frac{\partial}{\partial x}\left(\mu_{\mathrm{eff}}\frac{1}{r}\frac{\partial u}{\partial \theta}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\mu_{\mathrm{eff}}\left(\frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{w}{r}\right)\right)$ $+\frac{1}{r}\frac{\partial}{\partial \theta}\left(\mu_{\mathrm{eff}}\left(\frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{2v}{r}\right)\right) + \frac{1}{r}\mu_{\mathrm{eff}}\left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial r} - \frac{w}{r}\right) - \rho\frac{vw}{r}$

5 Schemes for the momentum equations

Some schemes for the momentum equations are discussed in this section. To summarize, the second order central difference scheme, the second order interpolation scheme and the TVD scheme[6] are used for the diffusion term, pressure gradient term and the convection term, respectively. For the source terms, both the Gaussian theorem and averaged volume integral are adopted.

5.1 The integral form of the governing equation

One of the key aspects of FVM is that the governing equation is first integrated over a

CV, then it the convection and diffusion terms are turned into surface integral by using the Gaussian theorem. For a general scalar variable ϕ, we have

$$\begin{gathered}\iiint_{V^{\mathrm{CV}}} \rho\left(\frac{\partial \phi}{\partial t}+\nabla\cdot(\mathbf{u}\phi)\right)\mathrm{d}v^{\mathrm{CV}} = \iiint_{V^{\mathrm{CV}}}\left(\nabla\cdot\left(\Gamma_{\phi}\nabla\phi\right)\right)\mathrm{d}v^{\mathrm{CV}} + \iiint_{V^{\mathrm{CV}}} s_{\phi}\mathrm{d}v^{\mathrm{CV}} \\ \Downarrow \\ \underbrace{\rho\iiint_{V^{\mathrm{CV}}}\frac{\partial\phi}{\partial t}\mathrm{d}v^{\mathrm{CV}}}_{(1)} + \underbrace{\rho\oint_{S^{\mathrm{CV}}}(\mathbf{u}\phi)\cdot\mathbf{n}\mathrm{d}s^{\mathrm{CV}}}_{(2)} = \underbrace{\oint_{S^{\mathrm{CV}}}\left(\Gamma_{\phi}\nabla\phi\right)\cdot\mathbf{n}\mathrm{d}s^{\mathrm{CV}}}_{(3)} + \underbrace{\iiint_{V^{\mathrm{CV}}} s_{\phi}\mathrm{d}v^{\mathrm{CV}}}_{(4)}\end{gathered} \tag{1.55}$$

where V^{CV} is the volume of the CV, v^{CV} is the infinitesimal volume element. S^{CV} and s^{CV} are the area and infinitesimal surface element of the faces of the CV. **n** is the unit normal vector of s^{CV}. The terms marked by (1) to (4) are the time changing rate, convection term, diffusion term and source term of the governing equation of ϕ. This equation becomes the momentum equation if ϕ is chosen to be a component of the velocity vector. And the pressure gradient term is contained in the source term. Let's ignore term (1) and keep (4) for later discussion. The first assignment is deriving the expressions for the schemes of (2) and (3).

In Eq. (1.55), the terms of (2) and (3) represent the net flux of physical properties. Generally for a vector **v**, the net flux across the faces of a CV is

$$\oint_{S^{\mathrm{CV}}}\mathbf{v}\cdot\mathbf{n}\mathrm{d}s^{\mathrm{CV}} \tag{1.56}$$

In coordinate (ξ, η, ζ), using the notations of Section 3, and based on Eq. (1.48), Eq. (1.56) can be written as

$$\begin{aligned}\oint_{S^{\mathrm{CV}}}\mathbf{v}\cdot\mathbf{n}\mathrm{d}s^{\mathrm{CV}} = & -\int_{\mathrm{b}}\mathbf{v}\cdot(\mathbf{n}\mathrm{d}s)^{\xi} + \int_{\mathrm{t}}\mathbf{v}\cdot(\mathbf{n}\mathrm{d}s)^{\xi} - \int_{\mathrm{s}}\mathbf{v}\cdot(\mathbf{n}\mathrm{d}s)^{\eta} + \int_{\mathrm{n}}\mathbf{v}\cdot(\mathbf{n}\mathrm{d}s)^{\eta} \\ & -\int_{\mathrm{e}}\mathbf{v}\cdot(\mathbf{n}\mathrm{d}s)^{\zeta} + \int_{\mathrm{w}}\mathbf{v}\cdot(\mathbf{n}\mathrm{d}s)^{\zeta}\end{aligned} \tag{1.57}$$

Using Eq. (1.35), we have

$$\begin{aligned}
&\oint_{S^{\mathrm{CV}}} \mathbf{v}\cdot\mathbf{n}\mathrm{d}s^{\mathrm{CV}} \\
&= -\int_{\mathrm{b}} \left(v^i\mathbf{g}_i\right)\cdot\left(\mathbf{n}\mathrm{d}s\right)^{\xi} + \int_{\mathrm{t}} \left(v^i\mathbf{g}_i\right)\cdot\left(\mathbf{n}\mathrm{d}s\right)^{\xi} - \int_{\mathrm{s}} \left(v^i\mathbf{g}_i\right)\cdot\left(\mathbf{n}\mathrm{d}s\right)^{\eta} + \int_{\mathrm{n}} \left(v^i\mathbf{g}_i\right)\cdot\left(\mathbf{n}\mathrm{d}s\right)^{\eta} \\
&\quad -\int_{\mathrm{e}} \left(v^i\mathbf{g}_i\right)\cdot\left(\mathbf{n}\mathrm{d}s\right)^{\zeta} + \int_{\mathrm{w}} \left(v^i\mathbf{g}_i\right)\cdot\left(\mathbf{n}\mathrm{d}s\right)^{\zeta} \\
&= -\int_{\mathrm{b}} v^1\mathbf{g}_1\cdot J^{\mathrm{tr}}\mathbf{g}^1\mathrm{d}\eta\mathrm{d}\zeta + \int_{\mathrm{t}} v^1\mathbf{g}_1\cdot J^{\mathrm{tr}}\mathbf{g}^1\mathrm{d}\eta\mathrm{d}\zeta - \int_{\mathrm{s}} v^2\mathbf{g}_2\cdot J^{\mathrm{tr}}\mathbf{g}^2\mathrm{d}\zeta\mathrm{d}\xi + \int_{\mathrm{n}} v^2\mathbf{g}_2\cdot J^{\mathrm{tr}}\mathbf{g}^2\mathrm{d}\zeta\mathrm{d}\xi \\
&\quad -\int_{\mathrm{e}} v^3\mathbf{g}_3\cdot J^{\mathrm{tr}}\mathbf{g}^3\mathrm{d}\xi\mathrm{d}\eta + \int_{\mathrm{w}} v^3\mathbf{g}_3\cdot J^{\mathrm{tr}}\mathbf{g}^3\mathrm{d}\xi\mathrm{d}\eta \\
&= -\int_{\mathrm{b}} v^1 J^{\mathrm{tr}}\mathrm{d}\eta\mathrm{d}\zeta + \int_{\mathrm{t}} v^1 J^{\mathrm{tr}}\mathrm{d}\eta\mathrm{d}\zeta - \int_{\mathrm{s}} v^2 J^{\mathrm{tr}}\mathrm{d}\zeta\mathrm{d}\xi + \int_{\mathrm{n}} v^2 J^{\mathrm{tr}}\mathrm{d}\zeta\mathrm{d}\xi \\
&\quad -\int_{\mathrm{e}} v^3 J^{\mathrm{tr}}\mathrm{d}\xi\mathrm{d}\eta + \int_{\mathrm{w}} v^3 J^{\mathrm{tr}}\mathrm{d}\xi\mathrm{d}\eta
\end{aligned}$$

(1.58)

Define general “area” of CV face to be

$$\begin{cases} A_{\mathrm{b}}^{\mathrm{tr}} = A_{\mathrm{t}}^{\mathrm{tr}} = \Delta\eta\Delta\zeta \\ A_{\mathrm{s}}^{\mathrm{tr}} = A_{\mathrm{n}}^{\mathrm{tr}} = \Delta\zeta\Delta\xi \\ A_{\mathrm{e}}^{\mathrm{tr}} = A_{\mathrm{w}}^{\mathrm{tr}} = \Delta\xi\Delta\eta \end{cases} \tag{1.59}$$

Then the discretized form of Eq. (1.58) is

$$\begin{aligned}\oint_{S^{\mathrm{CV}}} \mathbf{v}\cdot\mathbf{n}\mathrm{d}s^{\mathrm{CV}} \approx& -\left(v^1J^{\mathrm{tr}}\right)_{\mathrm{b}} A_{\mathrm{b}}^{\mathrm{tr}} + \left(v^1J^{\mathrm{tr}}\right)_{\mathrm{t}} A_{\mathrm{t}}^{\mathrm{tr}} - \left(v^2J^{\mathrm{tr}}\right)_{\mathrm{s}} A_{\mathrm{s}}^{\mathrm{tr}} + \left(v^2J^{\mathrm{tr}}\right)_{\mathrm{n}} A_{\mathrm{n}}^{\mathrm{tr}} \\ & -\left(v^3J^{\mathrm{tr}}\right)_{\mathrm{e}} A_{\mathrm{e}}^{\mathrm{tr}} + \left(v^3J^{\mathrm{tr}}\right)_{\mathrm{w}} A_{\mathrm{w}}^{\mathrm{tr}}\end{aligned} \tag{1.60}$$

Considering Table 1 and Eq. (1.41), Eq. (1.60) can be further modified as

$$\begin{aligned}\oint_{S^{\mathrm{CV}}} \mathbf{v}\cdot\mathbf{n}\mathrm{d}s^{\mathrm{CV}} \approx& -\left(rr_{,\eta}v_{\mathrm{x}}\right)_{\mathrm{b}} A_{\mathrm{b}}^{\mathrm{tr}} + \left(rr_{,\eta}v_{\mathrm{x}}\right)_{\mathrm{t}} A_{\mathrm{t}}^{\mathrm{tr}} \\ & -\left(rv_{\mathrm{r}} + \left(-r_{,\zeta}\right)v_{\theta}\right)_{\mathrm{s}} A_{\mathrm{s}}^{\mathrm{tr}} + \left(rv_{\mathrm{r}} + \left(-r_{,\zeta}\right)v_{\theta}\right)_{\mathrm{n}} A_{\mathrm{n}}^{\mathrm{tr}} \\ & -\left(r_{,\eta}v_{\theta}\right)_{\mathrm{e}} A_{\mathrm{e}}^{\mathrm{tr}} + \left(r_{,\eta}v_{\theta}\right)_{\mathrm{w}} A_{\mathrm{w}}^{\mathrm{tr}}\end{aligned} \tag{1.61}$$

Then we can define three components similar to the contravariant components

$$\begin{cases} V^{\mathrm{f1}} = J^{\mathrm{tr}} v^1 = & r r_{,\eta} v_{\mathrm{x}} \\ V^{\mathrm{f2}} = J^{\mathrm{tr}} v^2 = & r v_{\mathrm{r}} + \left(-r_{,\zeta}\right) v_{\theta} \\ V^{\mathrm{f3}} = J^{\mathrm{tr}} v^3 = & r_{,\eta} v_{\theta} \end{cases} \tag{1.62}$$

Insert Eq. (1.62) into (1.61)

$$\oint_{S^{\mathrm{CV}}} \mathbf{v} \cdot \mathbf{n} \mathrm{d}s^{\mathrm{CV}} \approx -V_{\mathrm{b}}^{\mathrm{f1}} A_{\mathrm{b}}^{\mathrm{tr}} + V_{\mathrm{t}}^{\mathrm{f1}} A_{\mathrm{t}}^{\mathrm{tr}} - V_{\mathrm{s}}^{\mathrm{f2}} A_{\mathrm{s}}^{\mathrm{tr}} + V_{\mathrm{n}}^{\mathrm{f2}} A_{\mathrm{n}}^{\mathrm{tr}} - V_{\mathrm{e}}^{\mathrm{f3}} A_{\mathrm{e}}^{\mathrm{tr}} + V_{\mathrm{w}}^{\mathrm{f3}} A_{\mathrm{w}}^{\mathrm{tr}} \tag{1.63}$$

In order to be distinguished from the contravariant components, Eq. (1.62) will be referred to as "face components". Eq. (1.64) gives the relation between the face components and the original components of vector **v**.

$$\begin{cases} v_{\mathrm{x}} = \dfrac{V^{\mathrm{f1}}}{r r_{,\eta}} \\ v_{\mathrm{r}} = \dfrac{1}{r}\left(V^{\mathrm{f2}} + \dfrac{r_{,\zeta}}{r_{,\eta}} V^{\mathrm{f3}}\right) \\ v_{\theta} = \dfrac{V^{\mathrm{f3}}}{r_{,\eta}} \end{cases} \tag{1.64}$$

5.2 Convection and diffusion terms

Let's start with the convection term in Eq. (1.55). Using Eq. (1.62) list every face component of velocity vector **u**.

$$U^{f1} = r r_{,\eta} u \tag{1.65}$$

$$U^{f2} = r v + \left(-r_{,\zeta}\right) w \tag{1.66}$$

$$U^{f3} = r_{,\eta} w \tag{1.67}$$

According to Eq. (1.63) and using Eq. (1.65) to (1.67) the convection term is

$$\rho \oint_{S^{\mathrm{CV}}} (\mathbf{u}\phi)\cdot\mathbf{n}\mathrm{d}s^{\mathrm{CV}} \approx \left(-\rho U^{f1}\phi A^{\mathrm{tr}}\right)_{\mathrm{b}} + \left(\rho U^{f1}\phi A^{\mathrm{tr}}\right)_{\mathrm{t}} + \left(-\rho U^{f2}\phi A^{\mathrm{tr}}\right)_{\mathrm{s}} + \left(\rho U^{f2}\phi A^{\mathrm{tr}}\right)_{\mathrm{n}} + \left(-\rho U^{f3}\phi A^{\mathrm{tr}}\right)_{\mathrm{e}} + \left(\rho U^{f3}\phi A^{\mathrm{tr}}\right)_{\mathrm{w}} \tag{1.68}$$

Define the flux of each CV face to be Eq. (1.69) to (1.71).

$$\dot{m}_{\mathrm{b}} = \left(-\rho U^{f1} A^{\mathrm{tr}}\right)_{\mathrm{b}}, \dot{m}_{\mathrm{t}} = \left(\rho U^{f1} A^{\mathrm{tr}}\right)_{\mathrm{t}} \tag{1.69}$$

$$\dot{m}_{\mathrm{s}} = \left(-\rho U^{f2} A^{\mathrm{tr}}\right)_{\mathrm{s}}, \dot{m}_{\mathrm{n}} = \left(\rho U^{f2} A^{\mathrm{tr}}\right)_{\mathrm{n}} \tag{1.70}$$

$$\dot{m}_{\mathrm{e}} = \left(-\rho U^{f3} A^{\mathrm{tr}}\right)_{\mathrm{e}}, \dot{m}_{\mathrm{w}} = \left(\rho U^{f3} A^{\mathrm{tr}}\right)_{\mathrm{w}} \tag{1.71}$$

Generally, let F be the flux of face q

$$F_{\mathrm{q}}^{\mathrm{m}} = \dot{m}_{\mathrm{q}} \tag{1.72}$$

where, q can be b, t, s, n, e and w. Then (1.68) could be written as

$$\rho \oint_{S^{\mathrm{CV}}} (\mathbf{u}\phi)\cdot\mathbf{n}\mathrm{d}s^{\mathrm{CV}} = \sum_{\mathrm{q}} F_{\mathrm{q}}^{\mathrm{m}} \phi_{\mathrm{q}} \tag{1.73}$$

Use TVD scheme to obtain the value of ϕ on the CV face[7].

$$\phi_{\mathrm{q}} = \phi_{\mathrm{qU}} + \frac{\psi_{\mathrm{FLF}}\left(r_{\mathrm{q}}^{\mathrm{TVD}}\right)}{2}\left(\phi_{\mathrm{qD}} - \phi_{\mathrm{qU}}\right) \tag{1.74}$$

where, ϕ_{qD} and ϕ_{qU} are the values at the center points of the downstream and upstream control volumes respect to face q. $r^{\mathrm{TVD}}{}_{\mathrm{q}}$ is expressed as

$$r_{\mathrm{q}}^{\mathrm{TVD}} = \frac{2\nabla\phi_{\mathrm{qU}}\cdot\mathbf{r}_{\mathrm{UD}}}{\phi_{\mathrm{qD}} - \phi_{\mathrm{qU}}} - 1 \tag{1.75}$$

where $\mathbf{r}_{\mathrm{UD}}$ is a vector that starts from the center point of the upstream CV and points to the center point of the downstream CV. The inner product between the gradient vector and **r** will be discussed later. At this point, we define the center point of the neighboring CV to be Q. In order to determine which of the two points, P and Q, should be the upstream point, the velocity vector of face q should be evaluated. It could be done by investigating the sign of the components of velocity **u** at point P expressed as Eq. (1.65) to (1.67). And a symbol α_q can be defined according to Table 3.

Table 3 α_q values.

face index	U^{f1}		U^{f2}		U^{f3}	
	< 0	> 0	< 0	> 0	< 0	> 0
b	1	0	--	--	--	--
t	0	1	--	--	--	--
s	--	--	1	0	--	--
n	--	--	0	1	--	--
e	--	--	--	--	1	0
w	--	--	--	--	0	1

-- means not relevant to α_q

For TVD scheme, ϕ_q can be expressed as

$$\begin{aligned}\phi_q &= \left(\alpha_q\phi_P + \left(1-\alpha_q\right)\phi_Q\right) \\ &+\frac{1}{2}\psi_{FLF}\left(r_q^{TVD}\right)\left[\left(\left(1-\alpha_q\right)\phi_P + \alpha_q\phi_Q\right) - \left(\alpha_q\phi_P + \left(1-\alpha_q\right)\phi_Q\right)\right]\end{aligned} \tag{1.76}$$

where ψ_{FLF} is the flux limiter function[6]. There are various types of flux limiter functions[8-12], and for this article I choose the one given by Van Albada[9].

$$\psi_{FLF}\left(r_q^{TVD}\right) = \frac{r_q^{TVD} + \left(r_q^{TVD}\right)^2}{1+\left(r_q^{TVD}\right)^2} \tag{1.77}$$

Then Eq. (1.73) can be written as

$$\rho\oint_{S^{CV}}\left(\mathbf{u}\phi\right)\cdot\mathbf{n}\mathrm{d}s^{CV} = \sum_q F_q^m\left(\alpha_q\phi_P + \left(1-\alpha_q\right)\phi_Q\right) - s^{TVD} \tag{1.78}$$

where

$$s^{TVD} = -\frac{1}{2}\sum_q F_q^m\psi_{FLF}\left(r_q^{TVD}\right)\left[\left(\left(1-\alpha_q\right)\phi_P + \alpha_q\phi_Q\right) - \left(\alpha_q\phi_P + \left(1-\alpha_q\right)\phi_Q\right)\right] \tag{1.79}$$

Let's move on to the diffusion term. The gradient vector of ϕ should also be written in the form of contravariant components. The gradient vector of ϕ in the original cylindrical coordinate system is

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{e}_{\mathrm{x}} + \frac{\partial\phi}{\partial r}\mathbf{e}_{\mathrm{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\mathbf{e}_{\theta} \tag{1.80}$$

Every face component (Eq. (1.62)) of vector $\nabla\phi$ is

$$\left(\nabla\phi\right)^{f1} = rr_{,\eta}\frac{\partial\phi}{\partial x} = rr_{,\eta}\frac{\partial\phi}{\partial\xi} = \frac{1}{J}rr_{,\eta}^{2}\frac{\partial\phi}{\partial\xi} \tag{1.81}$$

$$\begin{aligned}\left(\nabla\phi\right)^{f2} &= r\frac{\partial\phi}{\partial r} + \left(-r_{,\zeta}\right)\frac{1}{r}\frac{\partial\phi}{\partial\theta}\\ &= \frac{1}{J}r\frac{\partial\phi}{\partial\eta} + \left(-r_{,\zeta}\right)\frac{1}{r}\left(\frac{1}{J}\left(-r_{,\zeta}\frac{\partial\phi}{\partial\eta} + r_{,\eta}\frac{\partial\phi}{\partial\zeta}\right)\right)\\ &= \frac{1}{J}r\frac{\partial\phi}{\partial\eta} + \frac{1}{Jr}r_{,\zeta}^{2}\frac{\partial\phi}{\partial\eta} + \frac{1}{Jr}\left(-r_{,\zeta}r_{,\eta}\right)\frac{\partial\phi}{\partial\zeta}\\ &= \frac{1}{J}r\left(1+\frac{r_{,\zeta}^{2}}{r^{2}}\right)\frac{\partial\phi}{\partial\eta} + \frac{1}{J}r\left(-\frac{r_{,\eta}r_{,\zeta}}{r^{2}}\right)\frac{\partial\phi}{\partial\zeta}\end{aligned} \tag{1.82}$$

$$\begin{aligned}\left(\nabla\phi\right)^{f3} &= r_{,\eta}\frac{1}{r}\frac{\partial\phi}{\partial\theta}\\ &= r_{,\eta}\frac{1}{r}\frac{1}{J}\left(-r_{,\zeta}\frac{\partial\phi}{\partial\eta} + r_{,\eta}\frac{\partial\phi}{\partial\zeta}\right)\\ &= \frac{1}{J}r\left(-\frac{r_{,\eta}r_{,\zeta}}{r^{2}}\right)\frac{\partial\phi}{\partial\eta} + \frac{1}{J}r\frac{r_{,\eta}^{2}}{r^{2}}\frac{\partial\phi}{\partial\zeta}\end{aligned} \tag{1.83}$$

In fact, Eq. (1.81) to (1.83) take the same form with that obtained by Hong[13]. We can define a set of geometry parameters G

$$G_{1}^{\xi} = r_{,\eta}^{2} \tag{1.84}$$

$$G_{2}^{\eta} = 1+\frac{r_{,\zeta}^{2}}{r^{2}}, G_{3}^{\eta} = -\frac{r_{,\eta}r_{,\zeta}}{r^{2}} \tag{1.85}$$

$$G_{2}^{\zeta} = -\frac{r_{,\eta}r_{,\zeta}}{r^{2}} = G_{3}^{\eta}, G_{3}^{\zeta} = \frac{r_{,\eta}^{2}}{r^{2}} \tag{1.86}$$

Based on Eq. (1.63), we could revise the diffusion term of Eq. (1.55)

$$
\begin{aligned}
&\oint_{s^{\mathrm{CV}}} \left(\Gamma_{\phi}\nabla\phi\right)\cdot\mathbf{n}\mathrm{d}s^{\mathrm{CV}} \\
&\approx \left(-\Gamma_{\phi}\left(\nabla\phi\right)^{f1} A^{\mathrm{tr}}\right)_{\mathrm{b}} + \left(\Gamma_{\phi}\left(\nabla\phi\right)^{f1} A^{\mathrm{tr}}\right)_{\mathrm{t}} \\
&\quad + \left(-\Gamma_{\phi}\left(\nabla\phi\right)^{f2} A^{\mathrm{tr}}\right)_{\mathrm{s}} + \left(\Gamma_{\phi}\left(\nabla\phi\right)^{f2} A^{\mathrm{tr}}\right)_{\mathrm{n}} \\
&\quad + \left(-\Gamma_{\phi}\left(\nabla\phi\right)^{f3} A^{\mathrm{tr}}\right)_{\mathrm{e}} + \left(\Gamma_{\phi}\left(\nabla\phi\right)^{f3} A^{\mathrm{tr}}\right)_{\mathrm{w}} \\
&= \left(-\Gamma_{\phi}\frac{1}{J} rG_1^{\xi}\frac{\partial\phi}{\partial\xi} A^{\mathrm{tr}}\right)_{\mathrm{b}} + \left(\Gamma_{\phi}\frac{1}{J} rG_1^{\xi}\frac{\partial\phi}{\partial\xi} A^{\mathrm{tr}}\right)_{\mathrm{t}} \\
&\quad + \left(-\Gamma_{\phi}\left(\frac{1}{J} rG_2^{\eta}\frac{\partial\phi}{\partial\eta} + \frac{1}{J} rG_3^{\eta}\frac{\partial\phi}{\partial\zeta}\right) A^{\mathrm{tr}}\right)_{\mathrm{s}} + \left(\Gamma_{\phi}\left(\frac{1}{J} rG_2^{\eta}\frac{\partial\phi}{\partial\eta} + \frac{1}{J} rG_3^{\eta}\frac{\partial\phi}{\partial\zeta}\right) A^{\mathrm{tr}}\right)_{\mathrm{n}} \\
&\quad + \left(-\Gamma_{\phi}\left(\frac{1}{J} rG_2^{\zeta}\frac{\partial\phi}{\partial\eta} + \frac{1}{J} rG_3^{\zeta}\frac{\partial\phi}{\partial\zeta}\right) A^{\mathrm{tr}}\right)_{\mathrm{e}} + \left(\Gamma_{\phi}\left(\frac{1}{J} rG_2^{\zeta}\frac{\partial\phi}{\partial\eta} + \frac{1}{J} rG_3^{\zeta}\frac{\partial\phi}{\partial\zeta}\right) A^{\mathrm{tr}}\right)_{\mathrm{w}}
\end{aligned}
\tag{1.87}
$$

Rearrange the terms in Eq. (1.87).

$$
\begin{aligned}
&\oint_{s^{\mathrm{CV}}} \left(\Gamma_{\phi}\nabla\phi\right)\cdot\mathbf{n}\mathrm{d}s^{\mathrm{CV}} \\
&\approx \left(-\frac{\Gamma_{\phi}}{J} rG_1^{\xi}\frac{\partial\phi}{\partial\xi} A^{\mathrm{tr}}\right)_{\mathrm{b}} + \left(\frac{\Gamma_{\phi}}{J} rG_1^{\xi}\frac{\partial\phi}{\partial\xi} A^{\mathrm{tr}}\right)_{\mathrm{t}} \\
&\quad + \left(-\frac{\Gamma_{\phi}}{J} rG_2^{\eta}\frac{\partial\phi}{\partial\eta} A^{\mathrm{tr}}\right)_{\mathrm{s}} + \left(\frac{\Gamma_{\phi}}{J} rG_2^{\eta}\frac{\partial\phi}{\partial\eta} A^{\mathrm{tr}}\right)_{\mathrm{n}} \\
&\quad + \left(-\frac{\Gamma_{\phi}}{J} rG_3^{\zeta}\frac{\partial\phi}{\partial\zeta} A^{\mathrm{tr}}\right)_{\mathrm{e}} + \left(\frac{\Gamma_{\phi}}{J} rG_3^{\zeta}\frac{\partial\phi}{\partial\zeta} A^{\mathrm{tr}}\right)_{\mathrm{w}} + s^{\mathrm{trans}}
\end{aligned}
\tag{1.88}
$$

where s^{trans} is the source term that arises from the coordinate transform.

$$
\begin{aligned}
s^{\mathrm{trans}} &= \left(-\frac{\Gamma_{\phi}}{J} rG_3^{\eta}\frac{\partial\phi}{\partial\zeta} A^{\mathrm{tr}}\right)_{\mathrm{s}} + \left(\frac{\Gamma_{\phi}}{J} rG_3^{\eta}\frac{\partial\phi}{\partial\zeta} A^{\mathrm{tr}}\right)_{\mathrm{n}} \\
&\quad + \left(-\frac{\Gamma_{\phi}}{J} rG_2^{\zeta}\frac{\partial\phi}{\partial\eta} A^{\mathrm{tr}}\right)_{\mathrm{e}} + \left(\frac{\Gamma_{\phi}}{J} rG_2^{\zeta}\frac{\partial\phi}{\partial\eta} A^{\mathrm{tr}}\right)_{\mathrm{w}}
\end{aligned}
\tag{1.89}
$$

The detailed procedure to calculate s^{trans} will be discussed later (Eq. (1.115)). And now the diffusion term can be expressed as

$$\begin{aligned}&\oint_{S^{\mathrm{CV}}}\left(\Gamma_\phi\nabla\phi\right)\cdot\mathbf{n}\mathrm{d}s^{\mathrm{CV}}\\&\approx\left(\frac{\Gamma_\phi}{J}rG_1^{\xi}A^{\mathrm{tr}}\right)_{\mathrm{b}}\frac{\phi_{\mathrm{B}}-\phi_{\mathrm{P}}}{\delta_{\mathrm{PB}}}+\left(\frac{\Gamma_\phi}{J}rG_1^{\xi}A^{\mathrm{tr}}\right)_{\mathrm{t}}\frac{\phi_{\mathrm{T}}-\phi_{\mathrm{P}}}{\delta_{\mathrm{PT}}}\\&+\left(\frac{\Gamma_\phi}{J}rG_2^{\eta}A^{\mathrm{tr}}\right)_{\mathrm{s}}\frac{\phi_{\mathrm{S}}-\phi_{\mathrm{P}}}{\delta_{\mathrm{PS}}}+\left(\frac{\Gamma_\phi}{J}rG_2^{\eta}A^{\mathrm{tr}}\right)_{\mathrm{n}}\frac{\phi_{\mathrm{N}}-\phi_{\mathrm{P}}}{\delta_{\mathrm{PN}}}\\&+\left(\frac{\Gamma_\phi}{J}rG_3^{\zeta}A^{\mathrm{tr}}\right)_{\mathrm{e}}\frac{\phi_{\mathrm{E}}-\phi_{\mathrm{P}}}{\delta_{\mathrm{PE}}}+\left(\frac{\Gamma_\phi}{J}rG_3^{\zeta}A^{\mathrm{tr}}\right)_{\mathrm{w}}\frac{\phi_{\mathrm{W}}-\phi_{\mathrm{P}}}{\delta_{\mathrm{PW}}}+s^{\mathrm{trans}}\end{aligned}\tag{1.90}$$

where δ is the absolute value of the coordinate difference between point P and Q in coordinate system (ξ, η, ζ). Define parameter $D_{\mathrm{q,Q}}$

$$D_{\mathrm{q,Q}}=\left(\frac{\Gamma_\phi}{J}rGA^{\mathrm{tr}}\right)_{\mathrm{q}}\frac{1}{\delta_{\mathrm{PQ}}}\tag{1.91}$$

In Eq. (1.91), the values of G are listed in Table 4.

Table 4 *G* Values of control volume surfaces.

Face index	b	t	s	n	e	w
G	G^{ξ}_1		G^{η}_2		G^{ζ}_3	

Then Eq. (1.90) turns to be

$$\oint_{S^{\mathrm{CV}}}\left(\Gamma_\phi\nabla\phi\right)\cdot\mathbf{n}\mathrm{d}s^{\mathrm{CV}}\approx\sum_{\mathrm{q,Q}}D_{\mathrm{q,Q}}\left(\phi_{\mathrm{Q}}-\phi_{\mathrm{P}}\right)+s^{\mathrm{trans}}\tag{1.92}$$

Again, ignoring the time derivative term and using Eq. (1.78) and (1.92), the discretized form of Eq. (1.55) is

$$\sum_{\mathrm{q}}F_{\mathrm{q}}^{\mathrm{m}}\left(\alpha_{\mathrm{q}}\phi_{\mathrm{P}}+\left(1-\alpha_{\mathrm{q}}\right)\phi_{\mathrm{Q}}\right)-s^{\mathrm{TVD}}=\sum_{\mathrm{q,Q}}D_{\mathrm{q,Q}}\left(\phi_{\mathrm{Q}}-\phi_{\mathrm{P}}\right)+s^{\mathrm{trans}}+\iiint_{V^{\mathrm{CV}}}s_\phi\mathrm{d}v^{\mathrm{CV}}\tag{1.93}$$

Rearrange.

$$\begin{aligned}&\left(\sum_{\mathrm{q,Q}}\alpha_{\mathrm{q}}F_{\mathrm{q}}^{\mathrm{m}}+D_{\mathrm{q,Q}}\right)\phi_{\mathrm{P}}\\&=\sum_{\mathrm{q,Q}}\left(-\left(1-\alpha_{\mathrm{q}}\right)F_{\mathrm{q}}^{\mathrm{m}}+D_{\mathrm{q,Q}}\right)\phi_{\mathrm{Q}}+s^{\mathrm{TVD}}+s^{\mathrm{trans}}+\iiint_{V^{\mathrm{CV}}}s_\phi\mathrm{d}v^{\mathrm{CV}}\end{aligned}\tag{1.94}$$

where s^{TVD} and s^{trans} are known as the deferred correction terms[6, 14].

5.3 Source terms

First of all, it should be clear that in the original cylindrical coordinate system, the source terms of the momentum equations have different type of source terms. For any momentum equation, the source terms could be classified into three categories. Take the source terms of the momentum equation of θ direction as an example. The integral form of the source terms is

$$
\begin{aligned}
s_\theta &= \iiint_{V^{\mathrm{CV}}} s_\phi \mathrm{d}v^{\mathrm{CV}} \\
&= \underbrace{\iiint_{V^{\mathrm{CV}}} -\frac{1}{r}\frac{\partial p_{\mathrm{eff}}}{\partial \theta}\mathrm{d}v^{\mathrm{CV}}}_{s_\theta^{\mathrm{p}}} \\
&\quad + \underbrace{\iiint_{V^{\mathrm{CV}}} \left[\frac{\partial}{\partial x}\left(\mu_{\mathrm{eff}}\frac{1}{r}\frac{\partial u}{\partial \theta}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\mu_{\mathrm{eff}}\left(\frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{w}{r}\right)\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\mu_{\mathrm{eff}}\left(\frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{2v}{r}\right)\right)\right]\mathrm{d}v^{\mathrm{CV}}}_{s_\theta^{\mathrm{vis}}} \\
&\quad + \underbrace{\iiint_{V^{\mathrm{CV}}} \left(\frac{1}{r}\mu_{\mathrm{eff}}\left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial r} - \frac{w}{r}\right)\right)\mathrm{d}v^{\mathrm{CV}}}_{s_\theta^{\mathrm{cyl}}}
\end{aligned}
\tag{1.95}
$$

In this article, the source terms in a momentum equation are classified into three types denoting by s^{p}, s^{vis} and s^{cyl}. These three source terms are the pressure gradient source, viscosity source and special source term. Where the special source term, s^{cyl}, is the combination of special terms that directly associated to cylindrical coordinate system. The discretization method of each type of source term will be separately discussed.

5.3.1 Pressure gradient source term

As mentioned in the previous section, a second order scheme will be used to treat the pressure gradient source term. Like all the terms in Eq. (1.55), the pressure gradient source term, being one of the elements of the source term, takes the form of a integral over the CV. Generally, the scheme for the pressure gradient source term involves the application of the Gaussian theorem to transform the volume integral into surface integral. This requires that the pressure gradient of each momentum equation should be expressed in the form of divergence of a vector field. Using p instead of p_{eff} for convenience, we have

$$-\frac{\partial p}{\partial x}=-\nabla\cdot\mathbf{v}_{\mathrm{x}}^{\mathrm{p}},\quad \mathbf{v}_{\mathrm{x}}^{\mathrm{p}}=p\mathbf{e}_{\mathrm{x}}+0\mathbf{e}_{\mathrm{r}}+0\mathbf{e}_{\theta} \tag{1.96}$$

$$-\frac{\partial p}{\partial r}=-\nabla\cdot\mathbf{v}_{\mathrm{r}}^{\mathrm{p}}+\frac{p}{r},\quad \mathbf{v}_{\mathrm{r}}^{\mathrm{p}}=0\mathbf{e}_{\mathrm{x}}+p\mathbf{e}_{\mathrm{r}}+0\mathbf{e}_{\theta} \tag{1.97}$$

$$-\frac{1}{r}\frac{\partial p}{\partial \theta}=-\nabla\cdot\mathbf{v}_{\theta}^{\mathrm{p}},\quad \mathbf{v}_{\theta}^{\mathrm{p}}=0\mathbf{e}_{\mathrm{x}}+0\mathbf{e}_{\mathrm{r}}+p\mathbf{e}_{\theta} \tag{1.98}$$

The reason of Eq. (1.97) taking a different expression to the others is that the gradient operator has the special form, Eq. (1.99), in a cylindrical coordinate system.

$$\nabla\cdot\left(v_{\mathrm{x}}\mathbf{e}_{\mathrm{x}}+v_{\mathrm{r}}\mathbf{e}_{\mathrm{r}}+v_{\theta}\mathbf{e}_{\theta}\right)=\frac{\partial v_{\mathrm{x}}}{\partial x}+\frac{1}{r}\frac{\partial r v_{\mathrm{r}}}{\partial r}+\frac{1}{r}\frac{\partial v_{\theta}}{\partial \theta} \tag{1.99}$$

The volume integral of Eq. (1.96), (1.97) and (1.98) are needed. The detailed procedure is the same with that described in Section 5.2. Similar to the convection and diffusion terms, the face component of Eq. (1.96) to (1.98) are listed in Table 5, based on Eq. (1.63).

Table 5 Pressure components for volume integral.

i	$\left(V^{fi}\right)_{\mathrm{x}}^{\mathrm{p}}$	$\left(V^{fi}\right)_{\mathrm{r}}^{\mathrm{p}}$	$\left(V^{fi}\right)_{\theta}^{\mathrm{p}}$
1	$rr_{,\eta}p$	0	0
2	0	rp	$\left(-r_{,\zeta}\right)p$
3	0	0	$r_{,\eta}p$

From Table 5, the integrals of the pressure source term of each momentum equation are shown in Eq. (1.100) to (1.102).

$$s_{\mathrm{x}}^{\mathrm{p}}=\iiint_{V^{\mathrm{CV}}}-\frac{\partial p}{\partial x}\mathrm{d}v^{\mathrm{CV}}=\iiint_{V^{\mathrm{CV}}}-\nabla\cdot\mathbf{v}_{\mathrm{x}}^{\mathrm{p}}\mathrm{d}v^{\mathrm{CV}}\approx-\left(\left(-rr_{,\eta}pA^{\mathrm{tr}}\right)_{\mathrm{b}}+\left(rr_{,\eta}pA^{\mathrm{tr}}\right)_{\mathrm{t}}\right) \tag{1.100}$$

$$\begin{aligned} s_{\mathrm{r}}^{\mathrm{p}}&=\iiint_{V^{\mathrm{CV}}}-\frac{\partial p}{\partial r}\mathrm{d}v^{\mathrm{CV}}=\iiint_{V^{\mathrm{CV}}}-\nabla\cdot\mathbf{v}_{p,\mathrm{r}}\mathrm{d}v^{\mathrm{CV}}+\iiint_{V^{\mathrm{CV}}}\frac{p}{r}\mathrm{d}v^{\mathrm{CV}}\\ &\approx-\left(\left(-rpA^{\mathrm{tr}}\right)_{\mathrm{s}}+\left(rpA^{\mathrm{tr}}\right)_{\mathrm{n}}\right)+\iiint_{V^{\mathrm{CV}}_{(\xi,\eta,\zeta)}}\frac{p}{r}rJ\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta \end{aligned} \tag{1.101}$$

$$
\begin{aligned}
s_{\theta}^{\mathrm{p}} &= \iiint_{V^{\mathrm{CV}}} -\frac{1}{r}\frac{\partial p}{\partial \theta}\mathrm{d}v^{\mathrm{CV}} = \iiint_{V^{\mathrm{CV}}} -\nabla\cdot\mathbf{v}_{\theta}^{\mathrm{p}}\mathrm{d}v^{\mathrm{CV}} \\
&\approx -\left(\left(-\left(-r_{,\zeta}\right)pA^{\mathrm{tr}}\right)_{\mathrm{s}} + \left(\left(-r_{,\zeta}\right)pA^{\mathrm{tr}}\right)_{\mathrm{n}} + \left(-r_{,\eta}pA^{\mathrm{tr}}\right)_{\mathrm{e}} + \left(r_{,\eta}pA^{\mathrm{tr}}\right)_{\mathrm{w}}\right)
\end{aligned}
\tag{1.102}
$$

The subscript (ξ, η, ζ) in Eq. (1.101) means that the limits of the integral are defined in coordinate system (ξ, η, ζ). And this integral has to be approximated by averaged volume integral, Eq. (1.103).

$$
\iiint_{V_{(\xi,\eta,\zeta)}^{\mathrm{CV}}} \frac{p}{r} rJ\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta \approx pJV_{(\xi,\eta,\zeta)}^{\mathrm{CV,tr}} \tag{1.103}
$$

where $V_{(\xi,\eta,\zeta)}^{\mathrm{CV,tr}}$ is the general volume of the CV in coordinate system (ξ, η, ζ).

$$
V_{(\xi,\eta,\zeta)}^{\mathrm{CV,tr}} = \Delta\xi\Delta\eta\Delta\zeta \tag{1.104}
$$

Eq. (1.100) to (1.102) require that the pressure on face q of the CV to be calculated. This face pressure is defined as p_{q}. A second order central difference scheme, which is used in Ansys FLUENT[15], is applied in this article. The sketch in Fig. 5 shows two CVs straddling face q, and is served as an illustration of the scheme. This scheme is described by Eq. (1.105)

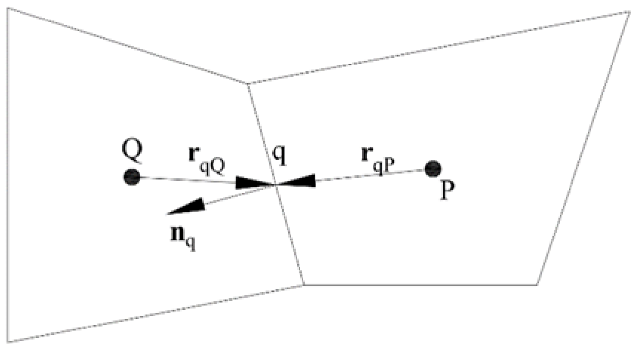

Fig. 5 The pressure on the face between two CVs, p_{q}.

$$
p_{\mathrm{q}} = \frac{1}{2}\left(p_{\mathrm{P}} + p_{\mathrm{Q}}\right) + \frac{1}{2}\left(\nabla p_{\mathrm{P}}\cdot\mathbf{r}_{\mathrm{qP}} + \nabla p_{\mathrm{Q}}\cdot\mathbf{r}_{\mathrm{qQ}}\right) \tag{1.105}
$$

where $\mathbf{r}_{\mathrm{qP}}$ and $\mathbf{r}_{\mathrm{qQ}}$ are vectors pointing to the center of face q, starting from point P and Q, respectively. ∇p_{P} and ∇p_{Q} are the pressure gradient vectors at point P and Q in coordinate system (ξ, η, ζ). Similar to the TVD scheme, the inner product between the pressure gradient vector and vector **r** will be discussed in following section. (Note that

$\mathbf{r}_{qP}$ and $\mathbf{r}_{qQ}$ will be approximated by $\mathbf{s}_{qP}$ and $\mathbf{s}_{qQ}$)

It should be pointed out that the symbol for pressure in the governing equations is p_{eff}. p_{eff} stands for "effective pressure". p_{eff} takes the form of

$$p_{\text{eff}} = p + \frac{2}{3}\rho k \tag{1.106}$$

In reality, the actual numeric value of the product of density ρ and turbulent kinetic energy k is relatively much smaller than the static pressure p. Therefore, the contribution of ρk to p is safely ignored. Thus

$$p_{\text{eff}} \approx p \tag{1.107}$$

For simplicity, we can use p to substitute p_{eff} in this article.

5.3.2 Viscous source terms

The viscous source terms in the momentum equations can be handled directly by Gaussian theorem. Take the term s^{vis}_{θ} in Eq. (1.95) for example and make up a dummy vector, Eq. (1.108).

$$\mathbf{v}_{\theta}^{\text{vis}} = \mu_{\text{eff}} \frac{1}{r}\frac{\partial u}{\partial \theta}\mathbf{e}_{\text{x}} + \mu_{\text{eff}}\left(\frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{w}{r}\right)\mathbf{e}_{\text{r}} + \mu_{\text{eff}}\left(\frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{2v}{r}\right)\mathbf{e}_{\theta} \tag{1.108}$$

The dummy vectors for the other two momentum equations can be written as

$$\mathbf{v}_{\text{x}}^{\text{vis}} = \mu_{\text{eff}}\frac{\partial u}{\partial x}\mathbf{e}_{\text{x}} + \mu_{\text{eff}}\frac{\partial v}{\partial x}\mathbf{e}_{\text{r}} + \mu_{\text{eff}}\frac{\partial w}{\partial x}\mathbf{e}_{\theta} \tag{1.109}$$

$$\mathbf{v}_{\text{r}}^{\text{vis}} = \mu_{\text{eff}}\frac{\partial u}{\partial r}\mathbf{e}_{\text{x}} + \mu_{\text{eff}}\frac{\partial v}{\partial r}\mathbf{e}_{\text{r}} + \mu_{\text{eff}}\left(\frac{\partial w}{\partial r} - \frac{w}{r}\right)\mathbf{e}_{\theta} \tag{1.110}$$

List the associated face components of Eq. (1.108) to (1.110) in Table 6 according to Eq. (1.62) .

Table 6 Contravariant components of the viscous source terms.

i	$\left(V^{fi}\right)_{\text{x}}^{\text{vis}}$	$\left(V^{fi}\right)_{\text{r}}^{\text{vis}}$	$\left(V^{fi}\right)_{\theta}^{\text{vis}}$
1	$rr_{,\eta}\mu_{\text{eff}}\frac{\partial u}{\partial x}$	$rr_{,\eta}\mu_{\text{eff}}\frac{\partial u}{\partial r}$	$rr_{,\eta}\mu_{\text{eff}}\frac{1}{r}\frac{\partial u}{\partial \theta}$

2	$r\mu_{\text{eff}}\dfrac{\partial v}{\partial x}$ $+\left(-r_{,\zeta}\right)\mu_{\text{eff}}\dfrac{\partial w}{\partial x}$	$r\mu_{\text{eff}}\dfrac{\partial v}{\partial r}$ $+\left(-r_{,\zeta}\right)\mu_{\text{eff}}\left(\dfrac{\partial w}{\partial r}-\dfrac{w}{r}\right)$	$r\mu_{\text{eff}}\left(\dfrac{1}{r}\dfrac{\partial v}{\partial \theta}-\dfrac{w}{r}\right)$ $+\left(-r_{,\zeta}\right)\mu_{\text{eff}}\left(\dfrac{1}{r}\dfrac{\partial w}{\partial \theta}+\dfrac{2v}{r}\right)$
3	$r_{,\eta}\mu_{\text{eff}}\dfrac{\partial w}{\partial x}$	$r_{,\eta}\mu_{\text{eff}}\left(\dfrac{\partial w}{\partial r}-\dfrac{w}{r}\right)$	$r_{,\eta}\mu_{\text{eff}}\left(\dfrac{1}{r}\dfrac{\partial w}{\partial \theta}+\dfrac{2v}{r}\right)$

The partial derivatives in Table 6 are expressed in there original cylindrical coordinate system. The transformed partial derivatives of arbitrary scalar ϕ are listed as

$$\frac{\partial \phi}{\partial x}=\frac{1}{J}r_{,\eta}\frac{\partial \phi}{\partial \xi} \tag{1.111}$$

$$\frac{\partial \phi}{\partial r}=\frac{1}{J}\frac{\partial \phi}{\partial \eta} \tag{1.112}$$

$$\frac{\partial \phi}{\partial \theta}=\frac{1}{J}\left(-r_{,\zeta}\frac{\partial \phi}{\partial \eta}+r_{,\eta}\frac{\partial \phi}{\partial \zeta}\right) \tag{1.113}$$

They are in fact the same with Eq. (1.16) to (1.18). They are repeated here to facilitate the usage of Table 6. If we use $\mathbf{v}^{\text{vis}}$ to represent any one of the viscous source terms, the integrals can be expressed in a universal form

$$\begin{aligned} s^{\text{vis}} &= \iiint_{V^{\text{CV}}} \nabla\cdot\mathbf{v}^{\text{vis}}\mathrm{d}v^{\text{CV}} = \oint_{S^{\text{CV}}} \mathbf{v}^{\text{vis}}\cdot\mathbf{n}\mathrm{d}s^{\text{CV}} \\ &\approx \left(-\left(V^{f1}\right)^{\text{vis}}A^{\text{tr}}\right)_{\text{b}}+\left(\left(V^{f1}\right)^{\text{vis}}A^{\text{tr}}\right)_{\text{t}} \\ &+\left(-\left(V^{f2}\right)^{\text{vis}}A^{\text{tr}}\right)_{\text{s}}+\left(\left(V^{f2}\right)^{\text{vis}}A^{\text{tr}}\right)_{\text{n}} \\ &+\left(-\left(V^{f3}\right)^{\text{vis}}A^{\text{tr}}\right)_{\text{e}}+\left(\left(V^{f3}\right)^{\text{vis}}A^{\text{tr}}\right)_{\text{w}} \end{aligned} \tag{1.114}$$

To obtain the physical property on face q, an interpolation method using values at neighboring center points is introduced. For general scalar value ϕ, let superscript int denote the interpolated value[16].

$$\phi_{\text{q}}^{\text{int}} = g_{\text{P}}\phi_{\text{P}} + g_{\text{Q}}\phi_{\text{Q}} \tag{1.115}$$

where g_{P} and g_{Q} are the interpolation weights. These weights are defined in a geometric sense, and are illustrated by Fig. 6 and Eq. (1.116).

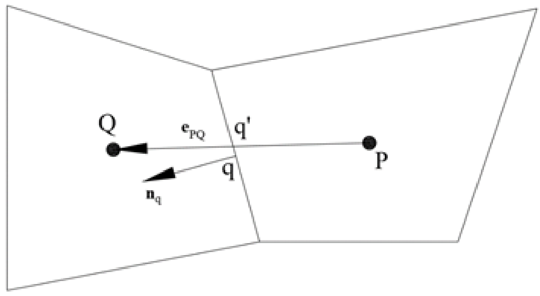

Fig. 6 Geometry weighted average.

$$\begin{cases} g_{\mathrm{Q}} = \dfrac{\delta_{\mathrm{Pq'}}}{\delta_{\mathrm{PQ}}} \\ g_{\mathrm{P}} = 1 - g_{\mathrm{Q}} \end{cases} \tag{1.116}$$

One thing should be cleared that in general case of grid structure, the line PQ intersects face q at a point q' other than the center of face q. However, in the current article, the grid is fully orthogonal after the coordinate transformation. Under this circumstance the point q' is the center of face q.

The value on face q in Eq. (1.114) turns to be

$$\left(\left(V^{fi}\right)^{\mathrm{vis}}\right)_{\mathrm{q}}^{\mathrm{int}} = \left(V^{fi}\right)_{\mathrm{q}}^{\mathrm{vis,int}} = g_{\mathrm{P}}\left(V^{fi}\right)_{\mathrm{P}}^{\mathrm{vis}} + g_{\mathrm{Q}}\left(V^{fi}\right)_{\mathrm{Q}}^{\mathrm{vis}}, \quad i = 1,2,3 \tag{1.117}$$

5.3.3 Special source terms

The special source term s^{cyl}, in Eq. (1.95) could be approximated by calculating the product (Eq. (1.104)) of volume and the physical property at the center point of the CV. In r direction

$$\begin{aligned} s_{\mathrm{r}}^{\mathrm{cyl}} &= \iiint_{V^{\mathrm{CV}}} \left(-\mu_{\mathrm{eff}} \frac{2}{r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{v}{r}\right) + \rho \frac{ww}{r}\right) \mathrm{d}v^{\mathrm{CV}} \\ &= \iiint_{V^{\mathrm{CV}}_{(\xi,\eta,\zeta)}} \left(-\mu_{\mathrm{eff}} \frac{2}{r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{v}{r}\right) + \rho \frac{ww}{r}\right) rJ\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta \\ &\approx \left(-\mu_{\mathrm{eff}} \frac{2}{r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{v}{r}\right) + \rho \frac{ww}{r}\right) rJV^{\mathrm{CV,tr}}_{(\xi,\eta,\zeta)} \end{aligned} \tag{1.118}$$

In the circumferential (θ) direction

$$
\begin{aligned}
s_{\theta}^{\mathrm{cyl}} &= \iiint\limits_{V^{\mathrm{CV}}} \left(\frac{1}{r} \mu_{\mathrm{eff}} \left(\frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial r} - \frac{w}{r} \right) - \rho \frac{vw}{r} \right) \mathrm{d} v^{\mathrm{CV}} \\
&= \iiint\limits_{V_{(\xi,\eta,\zeta)}^{\mathrm{CV}}} \left(\frac{1}{r} \mu_{\mathrm{eff}} \left(\frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial r} - \frac{w}{r} \right) - \rho \frac{vw}{r} \right) rJ \mathrm{d}\xi \mathrm{d}\eta \mathrm{d}\zeta \\
&\approx \left(\frac{1}{r} \mu_{\mathrm{eff}} \left(\frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial r} - \frac{w}{r} \right) - \rho \frac{vw}{r} \right) rJV_{(\xi,\eta,\zeta)}^{\mathrm{CV,tr}}
\end{aligned}
\tag{1.119}
$$

The x direction is ignored since $s^{\mathrm{cyl}}{}_{\mathrm{x}} = 0$. The partial derivatives in Eq. (1.118) and (1.119) could be handled by Eq. (1.111) to (1.113).

5.4 The general form of the momentum equations

Sum up all the source terms of momentum equations and write Eq. (1.120).

$$
S_{\phi} = \left(s^{\mathrm{TVD}} + s^{\mathrm{trans}} + s^{\mathrm{p}} + s^{\mathrm{vis}} + s^{\mathrm{cyl}} \right)_{\phi} \tag{1.120}
$$

The actual meaning of scalar ϕ changes according to each momentum equation. Define a new parameter a^{m}

$$
\begin{cases}
a_{\mathrm{P}}^{\mathrm{m}} = \sum\limits_{\mathrm{q,Q}} \left(\alpha_{\mathrm{q}} F_{\mathrm{q}}^{\mathrm{m}} + D_{\mathrm{q,Q}} \right) \\
a_{\mathrm{Q}}^{\mathrm{m}} = -\left(1 - \alpha_{\mathrm{q}} \right) F_{\mathrm{q}}^{\mathrm{m}} + D_{\mathrm{q,Q}}
\end{cases}
\tag{1.121}
$$

Based on Eq. (1.94) and (1.120), the discretized momentum equation could be expressed as general linear equation.

$$
a_{\mathrm{P}}^{\mathrm{m}} \phi_{\mathrm{P}} = \sum_{\mathrm{q,Q}} a_{\mathrm{Q}}^{\mathrm{m}} \phi_{\mathrm{Q}} + S_{\phi} \tag{1.122}
$$

When the time derivative is ignored and assume that pressure is know, the velocity components could be obtained by solving Eq. (1.122) in each coordinate direction. A fact that I want to point out is that $a^{\mathrm{m}}{}_{\mathrm{P}} \neq 0$.

It is worth to be noted that the parameters (a^{m}) of Eq. (1.122) for different momentum equation are the same.

Some first-order terms, e.g. s_{cyl} in Eq. (1.95), emerge in the source terms. Moukalled et. al.[16] recommended that the source term could be divided into zero-order and first-order terms.

$$S_\phi = S_\phi^0 + S_\phi^1 \phi \tag{1.123}$$

where ϕ is a scalar. The stability of the numerical solution process will be more stable if the term $S^1{}_\phi\phi$ is moved from the right hand side to the left. However, this treatment leads to different a^{m} parameters for each momentum equation and further scaling up the computational cost. Similar issues will be seen in the treatment of boundary conditions. Some other researcher, e.g. Ferziger and Perić[14], claimed that numerical stability can still be satisfactory if we do not follow Eq. (1.123). So does this article.

The linear equation of Eq. (1.122) is always solved by iterative solvers. For these solvers under-relaxation is needed for most of the cases. For the momentum equation, the implicit under-relaxation method is adopted. Let $\lambda^{\mathbf{u}}$ be the under-relaxation factor for momentum equation and modify Eq. (1.122).

$$\frac{a_{\mathrm{P}}^{\mathrm{m}}}{\lambda^{\mathbf{u}}} \phi_{\mathrm{P}} = \sum_{\mathrm{q,Q}} a_{\mathrm{Q}}^{\mathrm{m}} \phi_{\mathrm{Q}} + S_\phi + \frac{1-\lambda^{\mathbf{u}}}{\lambda^{\mathbf{u}}} a_{\mathrm{P}}^{\mathrm{m}} \phi_{\mathrm{P}}^{(n)} \tag{1.124}$$

where $\phi^{(n)}$ is the numeric solution of the previous iteration n. And

$$0 < \lambda^{\mathbf{u}} < 1 \tag{1.125}$$

6 The pressure-velocity coupling in co-located grid considering coordinate transformation

As mentioned in the previous sections, considering the additional complexity brought by the coordinate transformation the pressure-velocity coupling will be resolved by the SIMPLE method. Two key aspects of the SIMPLE method in co-located grid is the momentum interpolation and the equation of pressure correction.

6.1 Momentum interpolation

In co-located grid, one of the key issue of the SIMPLE algorithm is how to obtain the velocity on CV face. Let start from the discretized momentum equation, Eq. (1.124), at point P. Separate pressure source term s^{p} from S_ϕ. Take the x direction for example. s^{p} is approximated by

$$s_{\mathrm{x}}^{\mathrm{p}} = \iiint_{V^{\mathrm{CV}}} -\frac{\partial p}{\partial x} \mathrm{d}v^{\mathrm{CV}} \approx -\frac{\partial p}{\partial x} rJV_{(\xi,\eta,\zeta)}^{\mathrm{CV,tr}} = -\frac{1}{J} r_{,\eta} \frac{\partial p}{\partial \xi} rJV_{(\xi,\eta,\zeta)}^{\mathrm{CV,tr}} \tag{1.126}$$

Similarly, for the other two directions

$$s_{\mathrm{r}}^{\mathrm{p}} = \iiint_{V^{\mathrm{CV}}} -\frac{\partial p}{\partial r} \mathrm{d}v^{\mathrm{CV}} \approx -\frac{1}{J}\frac{\partial p}{\partial \eta} rJV_{(\xi,\eta,\zeta)}^{\mathrm{CV,tr}} \tag{1.127}$$

$$s_{\theta}^{\mathrm{p}} = \iiint_{V^{\mathrm{CV}}} -\frac{1}{r}\frac{\partial p}{\partial \theta} \mathrm{d}v^{\mathrm{CV}} \approx -\frac{1}{r}\frac{1}{J}\left(-r_{,\zeta}\frac{\partial p}{\partial \eta} + r_{,\eta}\frac{\partial p}{\partial \zeta}\right) rJV_{(\xi,\eta,\zeta)}^{\mathrm{CV,tr}} \tag{1.128}$$

Remember that $a^{\mathrm{m}}{}_{\mathrm{P}} \neq 0$, divide $a^{\mathrm{m}}{}_{\mathrm{P}}$ on both sides of Eq. (1.124). Then write Eq. (1.129) to (1.131) based on Eq. (1.126) to (1.128).

$$u_{\mathrm{P}} = H_{u,\mathrm{P}} - d_{\mathrm{P}}\left(r_{,\eta}\frac{\partial p}{\partial \xi}\right)_{\mathrm{P}} + \left(1-\lambda^{\mathbf{u}}\right)u_{\mathrm{P}}^{(n)} \tag{1.129}$$

$$v_{\mathrm{P}} = H_{v,\mathrm{P}} - d_{\mathrm{P}}\left(\frac{\partial p}{\partial \eta}\right)_{\mathrm{P}} + \left(1-\lambda^{\mathbf{u}}\right)v_{\mathrm{P}}^{(n)} \tag{1.130}$$

$$w_{\mathrm{P}} = H_{w,\mathrm{P}} - d_{\mathrm{P}}\left(\frac{1}{r}\left(-r_{,\zeta}\frac{\partial p}{\partial \eta} + r_{,\eta}\frac{\partial p}{\partial \zeta}\right)\right)_{\mathrm{P}} + \left(1-\lambda^{\mathbf{u}}\right)w_{\mathrm{P}}^{(n)} \tag{1.131}$$

where

$$H_{u,\mathrm{P}} = \frac{\lambda^{\mathbf{u}}\left(\sum_{\mathrm{q,Q}} a_{\mathrm{Q}}^{\mathrm{m}} u_{\mathrm{Q}} + S_u\right)}{a_{\mathrm{P}}^{\mathrm{m}}} \tag{1.132}$$

$$H_{v,\mathrm{P}} = \frac{\lambda^{\mathbf{u}}\left(\sum_{\mathrm{q,Q}} a_{\mathrm{Q}}^{\mathrm{m}} v_{\mathrm{Q}} + S_v\right)}{a_{\mathrm{P}}^{\mathrm{m}}} \tag{1.133}$$

$$H_{w,\mathrm{P}} = \frac{\lambda^{\mathbf{u}}\left(\sum_{\mathrm{q,Q}} a_{\mathrm{Q}}^{\mathrm{m}} v_{\mathrm{Q}} + S_w\right)}{a_{\mathrm{P}}^{\mathrm{m}}} \tag{1.134}$$

$$d_{\mathrm{P}} = \frac{\lambda^{\mathbf{u}} rV_{(\xi,\eta,\zeta)}^{\mathrm{CV,tr}}}{a_{\mathrm{P}}^{\mathrm{m}}} \tag{1.135}$$

$$S_u = \left(S_\phi\right)_{\mathrm{x}} - s_{\mathrm{x}}^{\mathrm{p}} \tag{1.136}$$

$$S_v = \left(S_\phi\right)_{\mathrm{r}} - s_{\mathrm{r}}^{\mathrm{p}} \tag{1.137}$$

$$S_w = \left(S_\phi\right)_\theta - s_\theta^{\mathrm{p}} \tag{1.138}$$

Note that Eq. (1.129) to (1.138) are defined at the center point of CV. The momentum equation on face q could be obtained by interpolation of the momentum equations on points P and Q, using Eq. (1.115). The interpolated equations on face q are

$$\begin{cases} u_{\mathrm{q}} = H_{u,\mathrm{q}}^{\mathrm{int}} - \left(d r_{,\eta}\right)_{\mathrm{q}}^{\mathrm{int}} \left(\dfrac{\partial p}{\partial \xi}\right)_{\mathrm{P}} + \left(1-\lambda^{\mathbf{u}}\right) u_{\mathrm{q}}^{(n)} \\ v_{\mathrm{q}} = H_{v,\mathrm{q}}^{\mathrm{int}} - \left(d\right)_{\mathrm{q}}^{\mathrm{int}} \left(\dfrac{\partial p}{\partial \eta}\right)_{\mathrm{P}} + \left(1-\lambda^{\mathbf{u}}\right) v_{\mathrm{q}}^{(n)} \\ w_{\mathrm{q}} = H_{w,\mathrm{q}}^{\mathrm{int}} - \left(d \dfrac{r_{,\eta}}{r}\right)_{\mathrm{q}}^{\mathrm{int}} \left(\dfrac{\partial p}{\partial \zeta}\right)_{\mathrm{q}} - \left(d \dfrac{-r_{,\zeta}}{r}\right)_{\mathrm{q}}^{\mathrm{int}} \left(\dfrac{\partial p}{\partial \eta}\right)_{\mathrm{q}} + \left(1-\lambda^{\mathbf{u}}\right) w_{\mathrm{q}}^{(n)} \end{cases} \tag{1.139}$$

Similar expressions could be found in other work[13, 17-20]. In fact Eq. (1.139) is the direct result of the momentum interpolation method of Rhie and Chow[21]. Further, based on the work of Moukalled et. al.[16], we can write

$$\begin{aligned} H_{u,\mathrm{q}}^{\mathrm{int}} &= g_{\mathrm{P}} \left(u_{\mathrm{P}} + d_{\mathrm{P}} \left(r_{,\eta} \frac{\partial p}{\partial \xi} \right)_{\mathrm{P}} - \left(1-\lambda^{\mathbf{u}}\right) u_{\mathrm{P}}^{(n)} \right) \\ &\quad + g_{\mathrm{Q}} \left(u_{\mathrm{Q}} + d_{\mathrm{Q}} \left(r_{,\eta} \frac{\partial p}{\partial \xi} \right)_{\mathrm{Q}} - \left(1-\lambda^{\mathbf{u}}\right) u_{\mathrm{Q}}^{(n)} \right) \\ H_{v,\mathrm{q}}^{\mathrm{int}} &= g_{\mathrm{P}} \left(v_{\mathrm{P}} + d_{\mathrm{P}} \left(\frac{\partial p}{\partial \eta} \right)_{\mathrm{P}} - \left(1-\lambda^{\mathbf{u}}\right) v_{\mathrm{P}}^{(n)} \right) \\ &\quad + g_{\mathrm{Q}} \left(v_{\mathrm{Q}} + d_{\mathrm{Q}} \left(\frac{\partial p}{\partial \eta} \right)_{\mathrm{Q}} - \left(1-\lambda^{\mathbf{u}}\right) v_{\mathrm{Q}}^{(n)} \right) \\ H_{w,\mathrm{q}}^{\mathrm{int}} &= g_{\mathrm{P}} \left(w_{\mathrm{P}} + d_{\mathrm{P}} \left(\frac{1}{r} \left(-r_{,\zeta} \frac{\partial p}{\partial \eta} + r_{,\eta} \frac{\partial p}{\partial \zeta} \right) \right)_{\mathrm{P}} - \left(1-\lambda^{\mathbf{u}}\right) w_{\mathrm{P}}^{(n)} \right) \\ &\quad + g_{\mathrm{Q}} \left(w_{\mathrm{Q}} + d_{\mathrm{Q}} \left(\frac{1}{r} \left(-r_{,\zeta} \frac{\partial p}{\partial \eta} + r_{,\eta} \frac{\partial p}{\partial \zeta} \right) \right)_{\mathrm{Q}} - \left(1-\lambda^{\mathbf{u}}\right) w_{\mathrm{Q}}^{(n)} \right) \end{aligned} \tag{1.140}$$

Thus

$$\begin{cases} H_{u,\mathrm{q}}^{\mathrm{int}} = u_{\mathrm{q}}^{\mathrm{int}} + \left(dr_{,\eta} \dfrac{\partial p}{\partial \xi} \right)_{\mathrm{q}}^{\mathrm{int}} - \left(1-\lambda^{\mathbf{u}}\right) u_{\mathrm{q}}^{(n),\mathrm{int}} \\ H_{v,\mathrm{q}}^{\mathrm{int}} = v_{\mathrm{q}}^{\mathrm{int}} + \left(d \dfrac{\partial p}{\partial \eta} \right)_{\mathrm{q}}^{\mathrm{int}} - \left(1-\lambda^{\mathbf{u}}\right) v_{\mathrm{q}}^{(n),\mathrm{int}} \\ H_{w,\mathrm{q}}^{\mathrm{int}} = w_{\mathrm{q}}^{\mathrm{int}} + \left(d \dfrac{1}{r} \left(-r_{,\zeta} \dfrac{\partial p}{\partial \eta} + r_{,\eta} \dfrac{\partial p}{\partial \zeta} \right) \right)_{\mathrm{q}}^{\mathrm{int}} - \left(1-\lambda^{\mathbf{u}}\right) w_{\mathrm{q}}^{(n),\mathrm{int}} \end{cases} \tag{1.141}$$

And we can write the following approximations.

$$\begin{cases} \left(dr_{,\eta} \dfrac{\partial p}{\partial \xi} \right)_{\mathrm{q}}^{\mathrm{int}} \approx \left(dr_{,\eta} \right)_{\mathrm{q}}^{\mathrm{int}} \left(\dfrac{\partial p}{\partial \xi} \right)_{\mathrm{q}}^{\mathrm{int}} \\ \left(d \dfrac{\partial p}{\partial \eta} \right)_{\mathrm{q}}^{\mathrm{int}} \approx \left(d \right)_{\mathrm{q}}^{\mathrm{int}} \left(\dfrac{\partial p}{\partial \eta} \right)_{\mathrm{q}}^{\mathrm{int}} \\ \left(d \dfrac{1}{r} \left(-r_{,\zeta} \dfrac{\partial p}{\partial \eta} + r_{,\eta} \dfrac{\partial p}{\partial \zeta} \right) \right)_{\mathrm{q}}^{\mathrm{int}} \approx \left(d \dfrac{-r_{,\zeta}}{r} \right)_{\mathrm{q}}^{\mathrm{int}} \left(\dfrac{\partial p}{\partial \eta} \right)_{\mathrm{q}}^{\mathrm{int}} + \left(d \dfrac{r_{,\eta}}{r} \right)_{\mathrm{q}}^{\mathrm{int}} \left(\dfrac{\partial p}{\partial \zeta} \right)_{\mathrm{q}}^{\mathrm{int}} \end{cases} \tag{1.142}$$

Insert Eq. (1.141) and (1.142) into Eq. (1.139).

$$\begin{aligned} u_{\mathrm{q}} &= u_{\mathrm{q}}^{\mathrm{int}} - \left(dr_{,\eta} \right)_{\mathrm{q}}^{\mathrm{int}} \left(\left(\dfrac{\partial p}{\partial \xi} \right)_{\mathrm{q}} - \left(\dfrac{\partial p}{\partial \xi} \right)_{\mathrm{q}}^{\mathrm{int}} \right) + \left(1-\lambda^{\mathbf{u}}\right) \left(u_{\mathrm{q}}^{(n)} - u_{\mathrm{q}}^{(n),\mathrm{int}} \right) \\ v_{\mathrm{q}} &= v_{\mathrm{q}}^{\mathrm{int}} - \left(d \right)_{\mathrm{q}}^{\mathrm{int}} \left(\left(\dfrac{\partial p}{\partial \eta} \right)_{\mathrm{q}} - \left(\dfrac{\partial p}{\partial \eta} \right)_{\mathrm{q}}^{\mathrm{int}} \right) + \left(1-\lambda^{\mathbf{u}}\right) \left(v_{\mathrm{q}}^{(n)} - v_{\mathrm{q}}^{(n),\mathrm{int}} \right) \\ w_{\mathrm{q}} &= w_{\mathrm{q}}^{\mathrm{int}} - \left(d \dfrac{r_{,\eta}}{r} \right)_{\mathrm{q}}^{\mathrm{int}} \left(\left(\dfrac{\partial p}{\partial \zeta} \right)_{\mathrm{q}} - \left(\dfrac{\partial p}{\partial \zeta} \right)_{\mathrm{q}}^{\mathrm{int}} \right) \\ &\quad - \left(d \dfrac{-r_{,\zeta}}{r} \right)_{\mathrm{q}}^{\mathrm{int}} \left(\left(\dfrac{\partial p}{\partial \eta} \right)_{\mathrm{q}} - \left(\dfrac{\partial p}{\partial \eta} \right)_{\mathrm{q}}^{\mathrm{int}} \right) + \left(1-\lambda^{\mathbf{u}}\right) \left(w_{\mathrm{q}}^{(n)} - w_{\mathrm{q}}^{(n),\mathrm{int}} \right) \end{aligned} \tag{1.143}$$

We need the covariant components of the velocity vector on CV face q. Note that we cannot use Eq. (1.62) but must apply Table 1.

$$\begin{aligned}\left(u^1\right)_{\mathrm{q}} &= \left(\frac{1}{J}r_{,\eta}\right)_{\mathrm{q}} u_{\mathrm{q}}^{\mathrm{int}} - \left(\frac{1}{J}r_{,\eta}\right)_{\mathrm{q}} \left(dr_{,\eta}\right)_{\mathrm{q}}^{\mathrm{int}} \left(\left(\frac{\partial p}{\partial \xi}\right)_{\mathrm{q}} - \left(\frac{\partial p}{\partial \xi}\right)_{\mathrm{q}}^{\mathrm{int}}\right) \\ &\quad + \left(1-\lambda^{\mathbf{u}}\right)\left(\frac{1}{J}r_{,\eta}\right)_{\mathrm{q}} \left(u_{\mathrm{q}}^{(n)} - u_{\mathrm{q}}^{(n),\mathrm{int}}\right) \\ &\approx \left(u^1\right)_{\mathrm{q}}^{\mathrm{int}} - \left(\frac{1}{J}r_{,\eta}\right)_{\mathrm{q}} \left(dr_{,\eta}\right)_{\mathrm{q}}^{\mathrm{int}} \left(\left(\frac{\partial p}{\partial \xi}\right)_{\mathrm{q}} - \left(\frac{\partial p}{\partial \xi}\right)_{\mathrm{q}}^{\mathrm{int}}\right) \\ &\quad + \left(1-\lambda^{\mathbf{u}}\right)\left(\left(u^1\right)_{\mathrm{q}}^{(n)} - \left(u^1\right)_{\mathrm{q}}^{(n),\mathrm{int}}\right)\end{aligned} \tag{1.144}$$

$$\begin{aligned}\left(u^2\right)_{\mathrm{q}} &= \left(\frac{1}{J}\right)_{\mathrm{q}} v_{\mathrm{q}}^{\mathrm{int}} - \left(\frac{1}{J}\right)_{\mathrm{q}} \left(d\right)_{\mathrm{q}}^{\mathrm{int}} \left(\left(\frac{\partial p}{\partial \eta}\right)_{\mathrm{q}} - \left(\frac{\partial p}{\partial \eta}\right)_{\mathrm{q}}^{\mathrm{int}}\right) + \left(1-\lambda^{\mathbf{u}}\right)\left(\frac{1}{J}\right)_{\mathrm{q}} \left(v_{\mathrm{q}}^{(n)} - v_{\mathrm{q}}^{(n),\mathrm{int}}\right) \\ &\quad + \left\{\left(\frac{-r_{,\zeta}}{Jr}\right)_{\mathrm{q}} w_{\mathrm{q}}^{\mathrm{int}} - \left(\frac{-r_{,\zeta}}{Jr}\right)_{\mathrm{q}} \left(d\,\frac{r_{,\eta}}{r}\right)_{\mathrm{q}}^{\mathrm{int}} \left(\left(\frac{\partial p}{\partial \zeta}\right)_{\mathrm{q}} - \left(\frac{\partial p}{\partial \zeta}\right)_{\mathrm{q}}^{\mathrm{int}}\right)\right. \\ &\quad \left. - \left(\frac{-r_{,\zeta}}{Jr}\right)_{\mathrm{q}} \left(d\,\frac{-r_{,\zeta}}{r}\right)_{\mathrm{q}}^{\mathrm{int}} \left(\left(\frac{\partial p}{\partial \eta}\right)_{\mathrm{q}} - \left(\frac{\partial p}{\partial \eta}\right)_{\mathrm{q}}^{\mathrm{int}}\right) + \left(1-\lambda^{\mathbf{u}}\right)\left(\frac{-r_{,\zeta}}{Jr}\right)_{\mathrm{q}} \left(w_{\mathrm{q}}^{(n)} - w_{\mathrm{q}}^{(n),\mathrm{int}}\right)\right\} \\ &\approx \left(u^2\right)_{\mathrm{q}}^{\mathrm{int}} - \left(\left(\frac{1}{J}\right)_{\mathrm{q}} \left(d\right)_{\mathrm{q}}^{\mathrm{int}} + \left(\frac{-r_{,\zeta}}{Jr}\right)_{\mathrm{q}} \left(d\,\frac{-r_{,\zeta}}{r}\right)_{\mathrm{q}}^{\mathrm{int}}\right)\left(\left(\frac{\partial p}{\partial \eta}\right)_{\mathrm{q}} - \left(\frac{\partial p}{\partial \eta}\right)_{\mathrm{q}}^{\mathrm{int}}\right) \\ &\quad + \left(1-\lambda^{\mathbf{u}}\right)\left(\left(u^2\right)_{\mathrm{q}}^{(n)} - \left(u^2\right)_{\mathrm{q}}^{(n),\mathrm{int}}\right) - \left(\frac{-r_{,\zeta}}{Jr}\right)_{\mathrm{q}} \left(d\,\frac{r_{,\eta}}{r}\right)_{\mathrm{q}}^{\mathrm{int}} \left(\left(\frac{\partial p}{\partial \zeta}\right)_{\mathrm{q}} - \left(\frac{\partial p}{\partial \zeta}\right)_{\mathrm{q}}^{\mathrm{int}}\right)\end{aligned}$$

(1.145)

$$\begin{aligned}\left(u^3\right)_{\mathrm{q}} &= \left(\frac{r_{,\eta}}{Jr}\right)_{\mathrm{q}} w_{\mathrm{q}}^{\mathrm{int}} - \left(\frac{r_{,\eta}}{Jr}\right)_{\mathrm{q}} \left(d\,\frac{r_{,\eta}}{r}\right)_{\mathrm{q}}^{\mathrm{int}} \left(\left(\frac{\partial p}{\partial \zeta}\right)_{\mathrm{q}} - \left(\frac{\partial p}{\partial \zeta}\right)_{\mathrm{q}}^{\mathrm{int}}\right) \\ &\quad - \left(\frac{r_{,\eta}}{Jr}\right)_{\mathrm{q}} \left(d\,\frac{-r_{,\zeta}}{r}\right)_{\mathrm{q}}^{\mathrm{int}} \left(\left(\frac{\partial p}{\partial \eta}\right)_{\mathrm{q}} - \left(\frac{\partial p}{\partial \eta}\right)_{\mathrm{q}}^{\mathrm{int}}\right) \\ &\quad + \left(1-\lambda^{\mathbf{u}}\right)\left(\frac{r_{,\eta}}{Jr}\right)_{\mathrm{q}} \left(w_{\mathrm{q}}^{(n)} - w_{\mathrm{q}}^{(n),\mathrm{int}}\right) \\ &\approx \left(u^3\right)_{\mathrm{q}}^{\mathrm{int}} - \left(\frac{r_{,\eta}}{Jr}\right)_{\mathrm{q}} \left(d\,\frac{r_{,\eta}}{r}\right)_{\mathrm{q}}^{\mathrm{int}} \left(\left(\frac{\partial p}{\partial \zeta}\right)_{\mathrm{q}} - \left(\frac{\partial p}{\partial \zeta}\right)_{\mathrm{q}}^{\mathrm{int}}\right) \\ &\quad + \left(1-\lambda^{\mathbf{u}}\right)\left(\left(u^3\right)_{\mathrm{q}}^{(n)} - \left(u^3\right)_{\mathrm{q}}^{(n),\mathrm{int}}\right) \\ &\quad - \left(\frac{r_{,\eta}}{Jr}\right)_{\mathrm{q}} \left(d\,\frac{-r_{,\zeta}}{r}\right)_{\mathrm{q}}^{\mathrm{int}} \left(\left(\frac{\partial p}{\partial \eta}\right)_{\mathrm{q}} - \left(\frac{\partial p}{\partial \eta}\right)_{\mathrm{q}}^{\mathrm{int}}\right)\end{aligned} \tag{1.146}$$

Similar results are also obtained by Yakinthos et. al.[22]and Qu et. al.[23]. To simplify the expressions, define new parameters for later use.

$$d_{1,\mathrm{q}}^{\xi} = \left(\frac{1}{J} r_{,\eta}\right)_{\mathrm{q}} \left(d r_{,\eta}\right)_{\mathrm{q}}^{\mathrm{int}} \tag{1.147}$$

$$d_{2,\mathrm{q}}^{\eta} = \left(\frac{1}{J}\right)_{\mathrm{q}} \left(d\right)_{\mathrm{q}}^{\mathrm{int}} + \left(\frac{-r_{,\zeta}}{Jr}\right)_{\mathrm{q}} \left(d \frac{-r_{,\zeta}}{r}\right)_{\mathrm{q}}^{\mathrm{int}} \tag{1.148}$$

$$d_{3,\mathrm{q}}^{\eta} = \left(\frac{-r_{,\zeta}}{Jr}\right)_{\mathrm{q}} \left(d \frac{r_{,\eta}}{r}\right)_{\mathrm{q}}^{\mathrm{int}} \tag{1.149}$$

$$d_{2,\mathrm{q}}^{\zeta} = \left(\frac{r_{,\eta}}{Jr}\right)_{\mathrm{q}} \left(d \frac{-r_{,\zeta}}{r}\right)_{\mathrm{q}}^{\mathrm{int}} \tag{1.150}$$

$$d_{3,\mathrm{q}}^{\zeta} = \left(\frac{r_{,\eta}}{Jr}\right)_{\mathrm{q}} \left(d \frac{r_{,\eta}}{r}\right)_{\mathrm{q}}^{\mathrm{int}} \tag{1.151}$$

Then Eq. (1.144) to (1.146) turn into

$$\left(u^1\right)_{\mathrm{q}} \approx \left(u^1\right)_{\mathrm{q}}^{\mathrm{int}} - d_{1,\mathrm{q}}^{\xi}\left(\left(\frac{\partial p}{\partial \xi}\right)_{\mathrm{q}} - \left(\frac{\partial p}{\partial \xi}\right)_{\mathrm{q}}^{\mathrm{int}}\right) + \left(1-\lambda^{\mathbf{u}}\right)\left(\left(u^1\right)_{\mathrm{q}}^{(n)} - \left(u^1\right)_{\mathrm{q}}^{(n),\mathrm{int}}\right) \tag{1.152}$$

$$\begin{aligned}\left(u^2\right)_{\mathrm{q}} \approx{}& \left(u^2\right)_{\mathrm{q}}^{\mathrm{int}} - d_{2,\mathrm{q}}^{\eta}\left(\left(\frac{\partial p}{\partial \eta}\right)_{\mathrm{q}} - \left(\frac{\partial p}{\partial \eta}\right)_{\mathrm{q}}^{\mathrm{int}}\right) \\ &+ \left(1-\lambda^{\mathbf{u}}\right)\left(\left(u^2\right)_{\mathrm{q}}^{(n)} - \left(u^2\right)_{\mathrm{q}}^{(n),\mathrm{int}}\right) - d_{3,\mathrm{q}}^{\eta}\left(\left(\frac{\partial p}{\partial \zeta}\right)_{\mathrm{q}} - \left(\frac{\partial p}{\partial \zeta}\right)_{\mathrm{q}}^{\mathrm{int}}\right)\end{aligned} \tag{1.153}$$

$$\begin{aligned}\left(u^3\right)_{\mathrm{q}} \approx{}& \left(u^3\right)_{\mathrm{q}}^{\mathrm{int}} - d_{3,\mathrm{q}}^{\zeta}\left(\left(\frac{\partial p}{\partial \zeta}\right)_{\mathrm{q}} - \left(\frac{\partial p}{\partial \zeta}\right)_{\mathrm{q}}^{\mathrm{int}}\right) \\ &+ \left(1-\lambda^{\mathbf{u}}\right)\left(\left(u^3\right)_{\mathrm{q}}^{(n)} - \left(u^3\right)_{\mathrm{q}}^{(n),\mathrm{int}}\right) - d_{2,\mathrm{q}}^{\zeta}\left(\left(\frac{\partial p}{\partial \eta}\right)_{\mathrm{q}} - \left(\frac{\partial p}{\partial \eta}\right)_{\mathrm{q}}^{\mathrm{int}}\right)\end{aligned} \tag{1.154}$$

6.2 Pressure correction equation

The SIMPLE algorithm is an iterative process. Let the velocity field and pressure field obtained in the previous iteration to be $\mathbf{u}^{(n)}=[u^{(n)},v^{(n)},w^{(n)}]^{\mathrm{T}}$ and $p^{(n)}$, respectively. n represents the n^{th} step of iteration.

6.2.1 The net mass flux and its correction

In the iteration of the SIMPLE algorithm, at iteration step $n + 1$, take $\mathbf{u}^{(n)}$ and $p^{(n)}$ from the previous iteration as the input and solve all the momentum equations in form of Eq. (1.124). The solution is an intermediate velocity field, defined as $\mathbf{u}^*$. Being the solution of the momentum equations, $\mathbf{u}^*$ does not necessarily satisfy the continuity equation. Then a correction of $\mathbf{u}^*$ is followed.

The continuity equation is treated in the same way as the momentum equation. Thus first take the integral form of the continuity equation then use the Gaussian theorem to transform the volume integral into surface integral. The surface integral is indeed the net mass flux of the CV. Using Eq. (1.63) and Eq. (1.69) to (1.71), the net mass flux is

$$\iiint_{V^{\mathrm{CV}}} \rho \nabla \cdot \mathbf{u} \mathrm{d} v^{\mathrm{CV}} = \oint_{S^{\mathrm{CV}}} \rho \mathbf{u} \cdot \mathbf{n} \mathrm{d} s^{\mathrm{CV}} \approx \sum_{\mathrm{q}} \dot{m}_{\mathrm{U,q}} \tag{1.155}$$

Using Eq. (1.62) we could obtain the face components of $\mathbf{u}^*$ and the real velocity field $\mathbf{u}$. These components are written as U^i and U^{i*}. U^i and U^{i*} satisfy the following relation.

$$U^i = U^{i*} + U^{i,\mathrm{c}} \tag{1.156}$$

where $U^{i,\mathrm{c}}$ is the velocity correction component. Apply the same definitions for the pressure field.

$$p = p^{(n)} + p^{\mathrm{c}} \tag{1.157}$$

From now on, the continuity equation becomes

$$\sum_{\mathrm{q}} \dot{m}_{\mathrm{q}} = \sum_{\mathrm{q}} \left(\dot{m}^* + \dot{m}^{\mathrm{c}} \right)_{\mathrm{q}} = 0 \tag{1.158}$$

where $\dot{m}^*_{\mathrm{q}}$ and $\dot{m}^{\mathrm{c}}_{\mathrm{q}}$ are the mass flux on CV face q, based on u^{i*} and $u^{i,\mathrm{c}}$. Now we apply the momentum interpolation and rewrite Eq. (1.152) to (1.154) by $\mathbf{u}^*$.

$$\begin{aligned} \left(u^{1*}\right)_{\mathrm{q}} &\approx \left(u^{1*}\right)_{\mathrm{q}}^{\mathrm{int}} - d_{1,\mathrm{q}}^{\xi} \left(\left(\frac{\partial p}{\partial \xi} \right)_{\mathrm{q}}^{(n)} - \left(\frac{\partial p}{\partial \xi} \right)_{\mathrm{q}}^{(n),\mathrm{int}} \right) \\ &+ \left(1 - \lambda^{\mathbf{u}}\right) \left(\left(u^1\right)_{\mathrm{q}}^{(n)} - \left(u^1\right)_{\mathrm{q}}^{(n),\mathrm{int}} \right) \end{aligned} \tag{1.159}$$

$$\begin{aligned}\left(u^{2*}\right)_{\mathrm{q}} &\approx \left(u^{2*}\right)_{\mathrm{q}}^{\mathrm{int}} - d_{2,\mathrm{q}}^{\eta}\left(\left(\frac{\partial p}{\partial \eta}\right)_{\mathrm{q}}^{(n)} - \left(\frac{\partial p}{\partial \eta}\right)_{\mathrm{q}}^{(n),\mathrm{int}}\right) \\ &+\left(1-\lambda^{\mathbf{u}}\right)\left(\left(u^{2}\right)_{\mathrm{q}}^{(n)} - \left(u^{2}\right)_{\mathrm{q}}^{(n),\mathrm{int}}\right) \underbrace{- d_{3,\mathrm{q}}^{\eta}\left(\left(\frac{\partial p}{\partial \zeta}\right)_{\mathrm{q}}^{(n)} - \left(\frac{\partial p}{\partial \zeta}\right)_{\mathrm{q}}^{(n),\mathrm{int}}\right)}_{\mathrm{cr}}\end{aligned} \tag{1.160}$$

$$\begin{aligned}\left(u^{3*}\right)_{\mathrm{q}} &\approx \left(u^{3*}\right)_{\mathrm{q}}^{\mathrm{int}} - d_{3,\mathrm{q}}^{\zeta}\left(\left(\frac{\partial p}{\partial \zeta}\right)_{\mathrm{q}}^{(n)} - \left(\frac{\partial p}{\partial \zeta}\right)_{\mathrm{q}}^{(n),\mathrm{int}}\right) \\ &+\left(1-\lambda^{\mathbf{u}}\right)\left(\left(u^{3}\right)_{\mathrm{q}}^{(n)} - \left(u^{3}\right)_{\mathrm{q}}^{(n),\mathrm{int}}\right) \underbrace{- d_{2,\mathrm{q}}^{\zeta}\left(\left(\frac{\partial p}{\partial \eta}\right)_{\mathrm{q}}^{(n)} - \left(\frac{\partial p}{\partial \eta}\right)_{\mathrm{q}}^{(n),\mathrm{int}}\right)}_{\mathrm{cr}}\end{aligned} \tag{1.161}$$

From Eq. (1.152) to (1.154) subtract Eq. (1.159) to (1.161), respectively. We obtain the following correction components.

$$\left(u^{1,\mathrm{c}}\right)_{\mathrm{q}} = \left(u^{1,\mathrm{c}}\right)_{\mathrm{q}}^{\mathrm{int}} - d_{1,\mathrm{q}}^{\xi}\left(\left(\frac{\partial p^{\mathrm{c}}}{\partial \xi}\right)_{\mathrm{q}} - \left(\frac{\partial p^{\mathrm{c}}}{\partial \xi}\right)_{\mathrm{q}}^{\mathrm{int}}\right) \tag{1.162}$$

$$\left(u^{2,\mathrm{c}}\right)_{\mathrm{q}} = \left(u^{2,\mathrm{c}}\right)_{\mathrm{q}}^{\mathrm{int}} - d_{2,\mathrm{q}}^{\eta}\left(\left(\frac{\partial p^{\mathrm{c}}}{\partial \eta}\right)_{\mathrm{q}} - \left(\frac{\partial p^{\mathrm{c}}}{\partial \eta}\right)_{\mathrm{q}}^{\mathrm{int}}\right) \underbrace{- d_{3,\mathrm{q}}^{\eta}\left(\left(\frac{\partial p^{\mathrm{c}}}{\partial \zeta}\right)_{\mathrm{q}} - \left(\frac{\partial p^{\mathrm{c}}}{\partial \zeta}\right)_{\mathrm{q}}^{\mathrm{int}}\right)}_{\mathrm{cr}} \tag{1.163}$$

$$\left(u^{3,\mathrm{c}}\right)_{\mathrm{q}} = \left(u^{3,\mathrm{c}}\right)_{\mathrm{q}}^{\mathrm{int}} - d_{3,\mathrm{q}}^{\zeta}\left(\left(\frac{\partial p^{\mathrm{c}}}{\partial \zeta}\right)_{\mathrm{q}} - \left(\frac{\partial p^{\mathrm{c}}}{\partial \zeta}\right)_{\mathrm{q}}^{\mathrm{int}}\right) \underbrace{- d_{2,\mathrm{q}}^{\zeta}\left(\left(\frac{\partial p^{\mathrm{c}}}{\partial \eta}\right)_{\mathrm{q}} - \left(\frac{\partial p^{\mathrm{c}}}{\partial \eta}\right)_{\mathrm{q}}^{\mathrm{int}}\right)}_{\mathrm{cr}} \tag{1.164}$$

The core approximation of the SIMPLE algorithm is that the velocity correction components are only depend on the pressure gradient, ignoring all the contribution from the velocity correction components of neighboring CVs. That is to say, ignoring all the terms marked by superscript “int” in Eq. (1.162) to (1.164). Here the terms marked by “cr” are also ignored as suggested by Hong[13]. Then Eq. (1.162) to (1.164) become

$$\left(u^{1,\mathrm{c}}\right)_{\mathrm{q}} \approx -d_{1,\mathrm{q}}^{\xi}\left(\frac{\partial p^{\mathrm{c}}}{\partial \xi}\right)_{\mathrm{q}} \tag{1.165}$$

$$\left(u^{2,\mathrm{c}}\right)_{\mathrm{q}} \approx -d_{2,\mathrm{q}}^{\eta}\left(\frac{\partial p^{\mathrm{c}}}{\partial \eta}\right)_{\mathrm{q}} \tag{1.166}$$

$$\left(u^{3,\mathrm{c}}\right)_{\mathrm{q}} \approx -d_{3,\mathrm{q}}^{\zeta}\left(\frac{\partial p^{\mathrm{c}}}{\partial \zeta}\right)_{\mathrm{q}} \tag{1.167}$$

Now Eq. (1.158) turns into

$$\sum_{\mathrm{q}} \dot{m}_{\mathrm{q}}^{\mathrm{c}} = -\sum_{\mathrm{q}} \dot{m}_{\mathrm{q}}^{*} \tag{1.168}$$

where

$$\begin{aligned}
&\sum_{\mathrm{q}} \dot{m}_{\mathrm{q}}^{*} \\
&= \left(-\rho u^{1*} J^{tr} A^{\mathrm{tr}}\right)_{\mathrm{b}} + \left(\rho u^{1*} J^{tr} A^{\mathrm{tr}}\right)_{\mathrm{t}} \\
&\quad + \left(-\rho u^{2*} J^{tr} A^{\mathrm{tr}}\right)_{\mathrm{s}} + \left(\rho u^{2*} J^{tr} A^{\mathrm{tr}}\right)_{\mathrm{n}} \\
&\quad + \left(-\rho u^{3*} J^{tr} A^{\mathrm{tr}}\right)_{\mathrm{e}} + \left(\rho u^{3*} J^{tr} A^{\mathrm{tr}}\right)_{\mathrm{w}}
\end{aligned} \tag{1.169}$$

$$\begin{aligned}
&\sum_{\mathrm{q}} \dot{m}_{\mathrm{q}}^{\mathrm{c}} \\
&= d_{1,\mathrm{b}}^{\xi}\left(\frac{\partial p^{\mathrm{c}}}{\partial \xi}\right)_{\mathrm{b}}\left(\rho J^{tr} A^{\mathrm{tr}}\right)_{\mathrm{b}} - d_{1,\mathrm{t}}^{\xi}\left(\frac{\partial p^{\mathrm{c}}}{\partial \xi}\right)_{\mathrm{t}}\left(\rho J^{tr} A^{\mathrm{tr}}\right)_{\mathrm{t}} \\
&\quad + d_{2,\mathrm{s}}^{\eta}\left(\frac{\partial p^{\mathrm{c}}}{\partial \eta}\right)_{\mathrm{s}}\left(\rho J^{tr} A^{\mathrm{tr}}\right)_{\mathrm{s}} - d_{2,\mathrm{n}}^{\eta}\left(\frac{\partial p^{\mathrm{c}}}{\partial \eta}\right)_{\mathrm{n}}\left(\rho J^{tr} A^{\mathrm{tr}}\right)_{\mathrm{n}} \\
&\quad + d_{3,\mathrm{e}}^{\zeta}\left(\frac{\partial p^{\mathrm{c}}}{\partial \zeta}\right)_{\mathrm{e}}\left(\rho J^{tr} A^{\mathrm{tr}}\right)_{\mathrm{e}} - d_{3,\mathrm{w}}^{\zeta}\left(\frac{\partial p^{\mathrm{c}}}{\partial \zeta}\right)_{\mathrm{w}}\left(\rho J^{tr} A^{\mathrm{tr}}\right)_{\mathrm{w}}
\end{aligned} \tag{1.170}$$

6.2.2 Pressure correction equation

After close inspection of the RHSs (right hand side) of Eq. (1.170), one could find out that the terms are very similar to the diffusion terms in the momentum equations. We could directly apply the similar strategy that we have just used to deal with the diffusion terms.

$$\left(\frac{\partial p^{\mathrm{c}}}{\partial \xi}\right)_{\mathrm{b}} = -\frac{p_{\mathrm{B}}^{\mathrm{c}} - p_{\mathrm{P}}^{\mathrm{c}}}{\delta_{\mathrm{PB}}} \tag{1.171}$$

$$\left(\frac{\partial p^{\mathrm{c}}}{\partial \xi}\right)_{\mathrm{t}} = \frac{p_{\mathrm{T}}^{\mathrm{c}} - p_{\mathrm{P}}^{\mathrm{c}}}{\delta_{\mathrm{PT}}} \tag{1.172}$$

$$\left(\frac{\partial p^{\mathrm{c}}}{\partial \eta}\right)_{\mathrm{s}} = -\frac{p_{\mathrm{S}}^{\mathrm{c}} - p_{\mathrm{P}}^{\mathrm{c}}}{\delta_{\mathrm{PS}}} \tag{1.173}$$

$$\left(\frac{\partial p^{\mathrm{c}}}{\partial \eta}\right)_{\mathrm{n}} = \frac{p_{\mathrm{N}}^{\mathrm{c}} - p_{\mathrm{P}}^{\mathrm{c}}}{\delta_{\mathrm{PN}}} \tag{1.174}$$

$$\left(\frac{\partial p^{\mathrm{c}}}{\partial \zeta}\right)_{\mathrm{e}} = -\frac{p_{\mathrm{E}}^{\mathrm{c}} - p_{\mathrm{P}}^{\mathrm{c}}}{\delta_{\mathrm{PE}}} \tag{1.175}$$

$$\left(\frac{\partial p^{\mathrm{c}}}{\partial \zeta}\right)_{\mathrm{w}} = \frac{p_{\mathrm{W}}^{\mathrm{c}} - p_{\mathrm{P}}^{\mathrm{c}}}{\delta_{\mathrm{PW}}} \tag{1.176}$$

Based on Eq. (1.169) to (1.176), we obtain the pressure correction equation.

$$a_{\mathrm{P}}^{\mathrm{c}} p_{\mathrm{P}}^{\mathrm{c}} + \sum_{\mathrm{Q}} a_{\mathrm{Q}}^{\mathrm{c}} p_{\mathrm{Q}}^{\mathrm{c}} = -\sum_{\mathrm{q}} \dot{m}_{\mathrm{q}}^{*} \tag{1.177}$$

where

$$a_{\mathrm{B}}^{\mathrm{c}} = -\frac{d_{1,\mathrm{b}}^{\xi}\left(\rho J^{tr} A^{\mathrm{tr}}\right)_{\mathrm{b}}}{\delta_{\mathrm{PB}}}, \quad a_{\mathrm{T}}^{\mathrm{c}} = -\frac{d_{1,\mathrm{t}}^{\xi}\left(\rho J^{tr} A^{\mathrm{tr}}\right)_{\mathrm{t}}}{\delta_{\mathrm{PT}}} \tag{1.178}$$

$$a_{\mathrm{S}}^{\mathrm{c}} = -\frac{d_{2,\mathrm{s}}^{\eta}\left(\rho J^{tr} A^{\mathrm{tr}}\right)_{\mathrm{s}}}{\delta_{\mathrm{PS}}}, \quad a_{\mathrm{N}}^{\mathrm{c}} = -\frac{d_{2,\mathrm{n}}^{\eta}\left(\rho J^{tr} A^{\mathrm{tr}}\right)_{\mathrm{n}}}{\delta_{\mathrm{PN}}} \tag{1.179}$$

$$a_{\mathrm{E}}^{\mathrm{c}} = -\frac{d_{3,\mathrm{e}}^{\zeta}\left(\rho J^{tr} A^{\mathrm{tr}}\right)_{\mathrm{e}}}{\delta_{\mathrm{PE}}}, \quad a_{\mathrm{W}}^{\mathrm{c}} = -\frac{d_{3,\mathrm{e}}^{\zeta}\left(\rho J^{tr} A^{\mathrm{tr}}\right)_{\mathrm{w}}}{\delta_{\mathrm{PW}}} \tag{1.180}$$

$$a_{\mathrm{P}}^{\mathrm{c}} = -\sum_{\mathrm{Q}} a_{\mathrm{Q}}^{\mathrm{c}} \tag{1.181}$$

Solve Eq. (1.177) numerically to obtain the p^{C} field. Then correct the intermediate velocity field by Eq. (1.182) to (1.184).

$$\left(u^{1}\right)_{\mathrm{P}}^{(n+1)} = \left(u^{1*}\right)_{\mathrm{P}} - d_{\mathrm{P}}\left(r_{,\eta} \frac{\partial p^{\mathrm{c}}}{\partial \xi}\right)_{\mathrm{P}} \tag{1.182}$$

$$\left(u^{2}\right)_{\mathrm{P}}^{(n+1)} = \left(u^{2*}\right)_{\mathrm{P}} - d_{\mathrm{P}}\left(\frac{\partial p^{\mathrm{c}}}{\partial \eta}\right)_{\mathrm{P}} \tag{1.183}$$

$$\left(u^{3}\right)_{\mathrm{P}}^{(n+1)} = \left(u^{3*}\right)_{\mathrm{P}} - d_{\mathrm{P}}\left(\frac{1}{r}\left(-r_{,\zeta} \frac{\partial p^{\mathrm{c}}}{\partial \eta} + r_{,\eta} \frac{\partial p^{\mathrm{c}}}{\partial \zeta}\right)\right)_{\mathrm{P}} \tag{1.184}$$

Note that the correction is done at the center point P. As for the pressure, use Eq. (1.185).

$$p^{(n+1)} = p^{(n)} + \lambda^{\mathrm{p}} p^{\mathrm{c}} \tag{1.185}$$

where λ^{p} is the under-relaxation factor for pressure correction. The values of the under-relaxation factors are listed in Table 7.

The above is the SIMPLE algorithm considering the coordinate transformation.

Table 7 Under-relaxation factors of various equations.

equation	under-relaxation factor	value
momentum eqs.	$\lambda^{\mathbf{u}}$	0.7
pressure correction eqs.	λ^{p}	0.3
eqs. of other scalars	λ^{s}	0.8

7 Other scalars

For incompressible turbulent flow. Some other scalars are still needed to be solved, e.g. the turbulence properties.

Since the equations for scalars are usually take the form of standard transport equation, the methodologies discussed so far are directly applicable. In other words, the convection, diffusion and source terms in the scalar transport equation are handle in the same way as for the momentum equations. (The under-relaxation method is also the same, and the recommended under-relaxation factor is listed in Table 7.)

8 Gradient reconstruction

One issue that is deliberately ignored is the calculation of the gradient of physical properties and the inner product between the gradient vector and general vector.

Gradient vector is needed by lots of procedures, e.g. the source terms and momentum interpolation. This article adopt the least-square gradient reconstruction[6, 16] method to calculate gradients. Since the underlying coordinate system is curvilinear, special treatment should be expected.

For an arbitrary scalar ϕ in the cylindrical coordinate system (x, r, θ), the relation among the values at the CV center P and the neighboring CV centers could be approximated by Eq. (1.186), which, in fact, is a Taylor expansion.

$$\phi_{\mathrm{Q}} \approx \phi_{\mathrm{P}} + (\nabla\phi)_{\mathrm{P}} \cdot (\mathbf{s}_{\mathrm{PQ}})_{\mathrm{P}} \tag{1.186}$$

where $\mathbf{s}_{\mathrm{PQ}}$ is actually a spatial curve expressed by Eq. (1.187).

$$\left(\mathbf{s}_{\mathrm{PQ}}\right)_{\mathrm{P}} = \left(\delta x_{\mathrm{PQ}}\mathbf{e}_{\mathrm{x}} + \delta r_{\mathrm{PQ}}\mathbf{e}_{\mathrm{r}} + r\delta\theta_{\mathrm{PQ}}\mathbf{e}_{\theta}\right)_{\mathrm{P}} \tag{1.187}$$

where δx_{PQ}, δr_{PQ} and $\delta\theta_{\mathrm{PQ}}$ are the three coordinate differences from point Q to P.

$$\begin{cases} \delta x_{\mathrm{PQ}} = x_{\mathrm{Q}} - x_{\mathrm{P}} \\ \delta r_{\mathrm{PQ}} = r_{\mathrm{Q}} - r_{\mathrm{P}} \\ \delta\theta_{\mathrm{PQ}} = \theta_{\mathrm{Q}} - \theta_{\mathrm{P}} \end{cases} \tag{1.188}$$

Note that Eq. (1.186) to (1.188) all have to be evaluated at the center point P of the CV under question. This means that the base vectors should be only specified at point P.

Now retrieve the expression of the gradient vector of scalar ϕ in cylindrical coordinate system.

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{e}_{\mathrm{x}} + \frac{\partial\phi}{\partial r}\mathbf{e}_{\mathrm{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\mathbf{e}_{\theta} \tag{1.189}$$

Its covariant components are

$$\left(\nabla\phi\right)_1 = \nabla\phi\cdot\mathbf{g}_1 = \frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial\xi} \tag{1.190}$$

$$\left(\nabla\phi\right)_2 = \nabla\phi\cdot\mathbf{g}_2 = r_{,\eta}\frac{\partial\phi}{\partial r} = \frac{\partial\phi}{\partial\eta} \tag{1.191}$$

$$\left(\nabla\phi\right)_3 = \nabla\phi\cdot\mathbf{g}_3 = r_{,\zeta}\frac{\partial\phi}{\partial r} + \frac{\partial\phi}{\partial\theta} = \frac{\partial\phi}{\partial\zeta} \tag{1.192}$$

Arrange the terms of Eq. (1.189) in the form of Eq. (1.36).

$$\nabla\phi = \frac{\partial\phi}{\partial\xi}\mathbf{g}^1 + \frac{\partial\phi}{\partial\eta}\mathbf{g}^2 + \frac{\partial\phi}{\partial\zeta}\mathbf{g}^3 \tag{1.193}$$

Now if we use the contravariant components to represent $\mathbf{s}_{\mathrm{PQ}}$, the inner product in Eq. (1.186) will become extremely easy to calculate. The contravariant components of $\mathbf{s}_{\mathrm{PQ}}$ are

$$\left(s_{\mathrm{PQ}}\right)_{\mathrm{P}}^1 = \left(\frac{1}{J}r_{,\eta}\delta x_{\mathrm{PQ}}\right)_{\mathrm{P}} \tag{1.194}$$

$$\left(s_{\mathrm{PQ}}\right)_{\mathrm{P}}^{2}=\left(\frac{1}{J}\delta r_{\mathrm{PQ}}+\frac{1}{J}\left(-r_{,\zeta}\right)\delta\theta_{\mathrm{PQ}}\right)_{\mathrm{P}} \tag{1.195}$$

$$\left(s_{\mathrm{PQ}}\right)_{\mathrm{P}}^{3}=\left(\frac{1}{J}r_{,\eta}\delta\theta_{\mathrm{PQ}}\right)_{\mathrm{P}} \tag{1.196}$$

Then

$$\left(\mathbf{s}_{\mathrm{PQ}}\right)_{\mathrm{P}}=\left(s_{\mathrm{PQ}}\right)_{\mathrm{P}}^{1}\left(\mathbf{g}_{1}\right)_{\mathrm{P}}+\left(s_{\mathrm{PQ}}\right)_{\mathrm{P}}^{2}\left(\mathbf{g}_{2}\right)_{\mathrm{P}}+\left(s_{\mathrm{PQ}}\right)_{\mathrm{P}}^{3}\left(\mathbf{g}_{3}\right)_{\mathrm{P}} \tag{1.197}$$

And thus, Eq. (1.186) becomes (the subscript P in some terms are ignored)

$$\phi_{\mathrm{Q}}=\phi_{\mathrm{P}}+\left(\nabla\phi\right)_{\mathrm{P}}\cdot\left(\mathbf{s}_{\mathrm{PQ}}\right)_{\mathrm{P}}=\phi_{\mathrm{P}}+\sum_{i}\left(\nabla\phi\right)_{i}\left(s_{\mathrm{PQ}}\right)^{i} \tag{1.198}$$

where i = 1, 2, 3. Move ϕ_{P} from the RHS to the LHS.

$$\left(\nabla\phi\right)_{1}\left(s_{\mathrm{PQ}}\right)^{1}+\left(\nabla\phi\right)_{2}\left(s_{\mathrm{PQ}}\right)^{2}+\left(\nabla\phi\right)_{3}\left(s_{\mathrm{PQ}}\right)^{3}=\phi_{\mathrm{Q}}-\phi_{\mathrm{P}} \tag{1.199}$$

Eq. (1.199) could be easily written in the form of a set of linear equations. Use the matrix form

$$\left[\mathbf{\Delta}\right]\left\{\nabla\phi\right\}=\left\{\Delta\phi\right\} \tag{1.200}$$

When there are m neighbors around the center CV, $\left\{\Delta\phi\right\}$ is a m by 1 column vector, [$\mathbf{\Delta}$] is a m by 3 matrix, $\left\{\nabla\phi\right\}$ is a 3 by 1 column vector (do not mix up with vector $\nabla\phi$). The above vectors and matrix are

$$\left\{\Delta\phi\right\}=\left\{\begin{matrix}\phi_{\mathrm{Q}1}-\phi_{\mathrm{P}}\\ \vdots\\ \phi_{\mathrm{Q}m}-\phi_{\mathrm{P}}\end{matrix}\right\}_{m\times1} \tag{1.201}$$

$$\left[\mathbf{\Delta}\right]=\begin{bmatrix}\left(s_{\mathrm{PQ}}\right)_{1}^{1} & \left(s_{\mathrm{PQ}}\right)_{1}^{2} & \left(s_{\mathrm{PQ}}\right)_{1}^{3}\\ \vdots & \vdots & \vdots\\ \left(s_{\mathrm{PQ}}\right)_{m}^{1} & \left(s_{\mathrm{PQ}}\right)_{m}^{2} & \left(s_{\mathrm{PQ}}\right)_{m}^{3}\end{bmatrix}_{m\times3} \tag{1.202}$$

$$\{\nabla\phi\} = \begin{Bmatrix} (\nabla\phi)_1 \\ (\nabla\phi)_2 \\ (\nabla\phi)_3 \end{Bmatrix} = \begin{Bmatrix} \left(\dfrac{\partial\phi}{\partial\xi}\right)_\mathrm{P} \\ \left(\dfrac{\partial\phi}{\partial\eta}\right)_\mathrm{P} \\ \left(\dfrac{\partial\phi}{\partial\zeta}\right)_\mathrm{P} \end{Bmatrix} \tag{1.203}$$

The set of linear equations Eq. (1.200) is generally overdetermined. To obtain a best approximated solution, the least-square method could be used. Multiply the transpose of [**Δ**] on both sides of Eq. (1.200).

$$[\mathbf{\Delta}]^\mathrm{T}[\mathbf{\Delta}]\{\nabla\phi\} = [\mathbf{\Delta}]^\mathrm{T}\{\Delta\phi\} \tag{1.204}$$

where $[\mathbf{\Delta}]^\mathrm{T}[\mathbf{\Delta}]$ is a 3 by 3 matrix. Usually $[\mathbf{\Delta}]^\mathrm{T}[\mathbf{\Delta}]$ is non-singular and its inverse matrix is relatively easy to solve. Then the solution of Eq. (1.204) is Eq. (1.205).

$$\{\nabla\phi\} = \left([\mathbf{\Delta}]^\mathrm{T}[\mathbf{\Delta}]\right)^{-1}[\mathbf{\Delta}]^\mathrm{T}\{\Delta\phi\} \tag{1.205}$$

Eq. (1.205) is the expression for solving gradient vector of scalar ϕ at CV center point P.

Here we can also summarize the method to calculate inner product between two vectors in the transformed curvilinear coordinate system. First find out the expression of the first vector with its covariant components. Then find the contravariant components of the second vector. Finally, just multiply the components and add them up like we did for $(\nabla\phi)_\mathrm{P}\cdot(\mathbf{s}_\mathrm{PQ})_\mathrm{P}$.

9 Boundary conditions

Only the issues associated with the coordinate transformation are discussed in this section, especially the wall treatment for momentum equation.

9.1 Wall treatment for momentum equation, non-slip wall

The non-slip wall boundary condition defines the wall velocity and wall shear stresses. On the CV face of wall, following the instructions of Moukalled et. al.[16], we have

$$\mathbf{F}_\mathrm{wall} = \boldsymbol{\tau}_\mathrm{wall}A_\mathrm{wall} \tag{1.206}$$

where $\mathbf{F}_{\text{wall}}$ is the force exert from the wall to the fluid in the CV, $\boldsymbol{\tau}_{\text{wall}}$ is the wall shear stress, A_{wall} is the area of the CV face. $\boldsymbol{\tau}_{\text{wall}}$ is calculated by Eq. (1.207).

$$\boldsymbol{\tau}_{\text{wall}} = -\mu_{\text{wall}} \frac{\partial \mathbf{u}_{\parallel}^{\text{diff}}}{\partial d_{\perp}} \approx -\mu_{\text{wall}} \left(\frac{\mathbf{u}_{\parallel}^{\text{diff}}}{d_{\perp}} \right)_{\text{P}} \tag{1.207}$$

where $\mathbf{u}_{\parallel}^{\text{diff}}$ is the velocity vector parallel to the wall, and is the difference between the fluid velocity and the wall velocity. $d_{\perp}$ is the vertical distance from a point to the wall. These two parameters should be calculated at CV center point P. So let's call them $\mathbf{u}_{\parallel,\text{P}}^{\text{diff}}$ and $d_{\perp,\text{P}}$.

$d_{\perp,\text{P}}$ is relatively easy to calculate. The calculation should be accomplished in coordinate system (ξ, η, ζ). Let $d_{\perp,\text{P}}^{\text{R}}$ and $d_{\perp,\text{P}}^{\text{S}}$ be the distances to the rotor and stator, respectively.

$$\begin{cases} d_{\perp,\text{P}}^{\text{R}} = \left| \left(r\mathbf{e}_{\text{r}} \right)_{\text{P}} - r_0 \mathbf{e}_{\text{y}} \right| - r^{\text{R}} \\ d_{\perp,\text{P}}^{\text{S}} = r^{\text{S}} - r_{\text{P}} \end{cases} \tag{1.208}$$

Let move to $\mathbf{u}_{\parallel,\text{P}}^{\text{diff}}$. First of all, calculate the difference between the fluid velocity and the wall velocity.

$$\mathbf{u}_{\text{P}}^{\text{diff}} = \mathbf{u}_{\text{P}} - \mathbf{u}_{\text{wall}} \tag{1.209}$$

It is notable that in curvilinear coordinate system, the vectors in Eq. (1.209) are using different base vectors. The base vectors of $\mathbf{u}_{\text{P}}$ is defined at point P while $\mathbf{u}_{\text{wall}}$ uses the base vectors defined at the CV face at wall.

$$\begin{aligned} \mathbf{u}_{\text{P}}^{\text{diff}} &= \mathbf{u}_{\text{P}} - \mathbf{u}_{\text{wall}} \\ &= \left(u\mathbf{e}_{\text{x}} + v\mathbf{e}_{\text{r}} + w\mathbf{e}_{\theta} \right)_{\text{P}} - \left(u\mathbf{e}_{\text{x}} + v\mathbf{e}_{\text{r}} + w\mathbf{e}_{\theta} \right)_{\text{wall}} \end{aligned} \tag{1.210}$$

In the current article, the ζ coordinate values for point P and the center point of CF face at wall are the same. Then Eq. (1.210) could be simplified as

$$\begin{aligned} &\mathbf{u}_{\text{P}}^{\text{diff}} = \mathbf{u}_{\text{P}} - \mathbf{u}_{\text{wall}} \\ &= \left(u_{\text{p}} \left(\mathbf{e}_{\text{x}} \right)_{\text{wall}} + v_{\text{p}} \left(\mathbf{e}_{\text{r}} \right)_{\text{wall}} + w_{\text{p}} \left(\mathbf{e}_{\theta} \right)_{\text{wall}} \right) - \left(u_{\text{wall}} \left(\mathbf{e}_{\text{x}} \right)_{\text{wall}} + v_{\text{wall}} \left(\mathbf{e}_{\text{r}} \right)_{\text{wall}} + w_{\text{wall}} \left(\mathbf{e}_{\theta} \right)_{\text{wall}} \right) \\ &= \left(u_{\text{p}} - u_{\text{wall}} \right) \left(\mathbf{e}_{\text{x}} \right)_{\text{wall}} + \left(v_{\text{p}} - v_{\text{wall}} \right) \left(\mathbf{e}_{\text{r}} \right)_{\text{wall}} + \left(w_{\text{p}} - w_{\text{wall}} \right) \left(\mathbf{e}_{\theta} \right)_{\text{wall}} \end{aligned}$$

(1.211)

And $\mathbf{u}_{\parallel,\mathrm{P}}^{\mathrm{diff}}$ can be expressed by

$$\mathbf{u}_{\parallel,\mathrm{P}}^{\mathrm{diff}} = \mathbf{u}_{\mathrm{P}}^{\mathrm{diff}} - \left(\mathbf{u}_{\mathrm{P}}^{\mathrm{diff}} \cdot \mathbf{n}_{\mathrm{wall}}\right)\mathbf{n}_{\mathrm{wall}} \tag{1.212}$$

For rotor and stator surfaces, $\mathbf{n}_{\mathrm{wall}}$ can be expressed based on Eq. (1.47). Take rotor surface for example.

$$\mathbf{n}_{\mathrm{wall}}^{\mathrm{R}} = \left(\frac{-J^{tr}\mathbf{g}^2}{\left|J^{tr}\mathbf{g}^2\right|}\right)_{\mathrm{wall}} = \left(\frac{-r}{\sqrt{r^2+r_{,\zeta}^2}}\mathbf{e}_{\mathrm{r}}\right)_{\mathrm{wall}} + \left(\frac{r_{,\zeta}}{\sqrt{r^2+r_{,\zeta}^2}}\mathbf{e}_{\theta}\right)_{\mathrm{wall}} \tag{1.213}$$

Thus

$$\begin{aligned}
&\left(\left(\mathbf{u}_{\mathrm{P}}^{\mathrm{diff}} \cdot \mathbf{n}_{\mathrm{wall}}\right)\mathbf{n}_{\mathrm{wall}}\right)^{\mathrm{R}} \\
&= \left(\left(v_{\mathrm{p}} - v_{\mathrm{wall}}\right)\frac{-r}{\sqrt{r^2+r_{,\zeta}^2}} + \left(w_{\mathrm{p}} - w_{\mathrm{wall}}\right)\frac{r_{,\zeta}}{\sqrt{r^2+r_{,\zeta}^2}}\right)_{\mathrm{wall}}^{\mathrm{R}} \left(\left(\frac{-r}{\sqrt{r^2+r_{,\zeta}^2}}\mathbf{e}_{\mathrm{r}}\right)_{\mathrm{wall}}^{\mathrm{R}} + \left(\frac{r_{,\zeta}}{\sqrt{r^2+r_{,\zeta}^2}}\mathbf{e}_{\theta}\right)_{\mathrm{wall}}^{\mathrm{R}}\right) \\
&= \left(\left(\left(v_{\mathrm{p}} - v_{\mathrm{wall}}\right)\frac{r^2}{r^2+r_{,\zeta}^2} + \left(w_{\mathrm{p}} - w_{\mathrm{wall}}\right)\frac{-rr_{,\zeta}}{r^2+r_{,\zeta}^2}\right)\mathbf{e}_{\mathrm{r}}\right)_{\mathrm{wall}}^{\mathrm{R}} \\
&+ \left(\left(\left(v_{\mathrm{p}} - v_{\mathrm{wall}}\right)\frac{-rr_{,\zeta}}{r^2+r_{,\zeta}^2} + \left(w_{\mathrm{p}} - w_{\mathrm{wall}}\right)\frac{r_{,\zeta}^2}{r^2+r_{,\zeta}^2}\right)\mathbf{e}_{\theta}\right)_{\mathrm{wall}}^{\mathrm{R}}
\end{aligned} \tag{1.214}$$

Similar expressions could be obtained for the stator surface. now insert Eq. (1.214) into Eq. (1.212), and ignore superscript R.

$$
\begin{aligned}
\mathbf{u}_{\parallel,\mathrm{P}}^{\mathrm{diff}} &= \mathbf{u}_{\mathrm{P}}^{\mathrm{diff}} - \left(\mathbf{u}_{\mathrm{P}}^{\mathrm{diff}} \cdot \mathbf{n}_{\mathrm{wall}}\right)\mathbf{n}_{\mathrm{wall}} \\
&= \left(u_{\mathrm{p}} - u_{\mathrm{wall}}\right)\left(\mathbf{e}_{\mathrm{x}}\right)_{\mathrm{wall}} \\
&+ \left(v_{\mathrm{p}} - v_{\mathrm{wall}}\right)\left(\mathbf{e}_{\mathrm{r}}\right)_{\mathrm{wall}} - \left(\left(\left(v_{\mathrm{p}} - v_{\mathrm{wall}}\right)\frac{r^2}{r^2 + r_{,\zeta}^2} + \left(w_{\mathrm{p}} - w_{\mathrm{wall}}\right)\frac{-rr_{,\zeta}}{r^2 + r_{,\zeta}^2}\right)\mathbf{e}_{\mathrm{r}}\right)_{\mathrm{wall}} \\
&+ \left(w_{\mathrm{p}} - w_{\mathrm{wall}}\right)\left(\mathbf{e}_{\theta}\right)_{\mathrm{wall}} - \left(\left(\left(v_{\mathrm{p}} - v_{\mathrm{wall}}\right)\frac{-rr_{,\zeta}}{r^2 + r_{,\zeta}^2} + \left(w_{\mathrm{p}} - w_{\mathrm{wall}}\right)\frac{r_{,\zeta}^2}{r^2 + r_{,\zeta}^2}\right)\mathbf{e}_{\theta}\right)_{\mathrm{wall}} \\
&= \left(u_{\mathrm{p}} - u_{\mathrm{wall}}\right)\left(\mathbf{e}_{\mathrm{x}}\right)_{\mathrm{wall}} \\
&+ \left(\left(\left(v_{\mathrm{p}} - v_{\mathrm{wall}}\right)\frac{r_{,\zeta}^2}{r^2 + r_{,\zeta}^2} + \left(w_{\mathrm{p}} - w_{\mathrm{wall}}\right)\frac{rr_{,\zeta}}{r^2 + r_{,\zeta}^2}\right)\mathbf{e}_{\mathrm{r}}\right)_{\mathrm{wall}} \\
&+ \left(\left(\left(v_{\mathrm{p}} - v_{\mathrm{wall}}\right)\frac{rr_{,\zeta}}{r^2 + r_{,\zeta}^2} + \left(w_{\mathrm{p}} - w_{\mathrm{wall}}\right)\frac{r^2}{r^2 + r_{,\zeta}^2}\right)\mathbf{e}_{\theta}\right)_{\mathrm{wall}}
\end{aligned}
\tag{1.215}
$$

Insert Eq. (1.215) into Eq. (1.207) and further insert the result into Eq. (1.206). Then we get $\mathbf{F}_{\mathrm{wall}}$.

$$
\begin{aligned}
\mathbf{F}_{\mathrm{wall}} &\approx -\mu_{\mathrm{wall}}\frac{\mathbf{u}_{\parallel,\mathrm{P}}^{\mathrm{diff}}}{d_{\perp,\mathrm{P}}}A_{\mathrm{wall}} \\
&= \frac{-\mu_{\mathrm{wall}}A_{\mathrm{wall}}}{d_{\perp,\mathrm{P}}}\left(u_{\mathrm{p}} - u_{\mathrm{wall}}\right)\left(\mathbf{e}_{\mathrm{x}}\right)_{\mathrm{wall}} \\
&+ \frac{-\mu_{\mathrm{wall}}A_{\mathrm{wall}}}{d_{\perp,\mathrm{P}}}\left(\left(\left(v_{\mathrm{p}} - v_{\mathrm{wall}}\right)\frac{r_{,\zeta}^2}{r^2 + r_{,\zeta}^2} + \left(w_{\mathrm{p}} - w_{\mathrm{wall}}\right)\frac{rr_{,\zeta}}{r^2 + r_{,\zeta}^2}\right)\mathbf{e}_{\mathrm{r}}\right)_{\mathrm{wall}} \\
&+ \frac{-\mu_{\mathrm{wall}}A_{\mathrm{wall}}}{d_{\perp,\mathrm{P}}}\left(\left(\left(v_{\mathrm{p}} - v_{\mathrm{wall}}\right)\frac{rr_{,\zeta}}{r^2 + r_{,\zeta}^2} + \left(w_{\mathrm{p}} - w_{\mathrm{wall}}\right)\frac{r^2}{r^2 + r_{,\zeta}^2}\right)\mathbf{e}_{\theta}\right)_{\mathrm{wall}} \\
&= -a_{\mathrm{wall}}^{\mathrm{m,x}}u_{\mathrm{p}}\left(\mathbf{e}_{\mathrm{x}}\right)_{\mathrm{wall}} - a_{\mathrm{wall}}^{\mathrm{m,r}}v_{\mathrm{p}}\left(\mathbf{e}_{\mathrm{r}}\right)_{\mathrm{wall}} - a_{\mathrm{wall}}^{\mathrm{m},\theta}w_{\mathrm{p}}\left(\mathbf{e}_{\theta}\right)_{\mathrm{wall}} \\
&+ s_{\mathrm{wall}}^{\mathrm{m,x}}\left(\mathbf{e}_{\mathrm{x}}\right)_{\mathrm{wall}} + s_{\mathrm{wall}}^{\mathrm{m,r}}\left(\mathbf{e}_{\mathrm{r}}\right)_{\mathrm{wall}} + s_{\mathrm{wall}}^{\mathrm{m},\theta}\left(\mathbf{e}_{\theta}\right)_{\mathrm{wall}}
\end{aligned}
\tag{1.216}
$$

where

$$
\begin{cases}
a_{\text{wall}}^{\text{m,x}} = \dfrac{\mu_{\text{wall}} A_{\text{wall}}}{d_{\perp,\text{P}}} \\
a_{\text{wall}}^{\text{m,r}} = \dfrac{\mu_{\text{wall}} A_{\text{wall}}}{d_{\perp,\text{P}}} \left(\dfrac{r_{,\zeta}^2}{r^2 + r_{,\zeta}^2} \right)_{\text{wall}} \\
a_{\text{wall}}^{\text{m},\theta} = \dfrac{\mu_{\text{wall}} A_{\text{wall}}}{d_{\perp,\text{P}}} \left(\dfrac{r^2}{r^2 + r_{,\zeta}^2} \right)_{\text{wall}} \\
s_{\text{wall}}^{\text{m,x}} = \dfrac{\mu_{\text{wall}} A_{\text{wall}}}{d_{\perp,\text{P}}} u_{\text{wall}} \\
s_{\text{wall}}^{\text{m,r}} = \dfrac{\mu_{\text{wall}} A_{\text{wall}}}{d_{\perp,\text{P}}} \left(v_{\text{wall}} \dfrac{r_{,\zeta}^2}{r^2 + r_{,\zeta}^2} + \left(w_{\text{p}} - w_{\text{wall}} \right) \dfrac{-r r_{,\zeta}}{r^2 + r_{,\zeta}^2} \right)_{\text{wall}} \\
s_{\text{wall}}^{\text{m},\theta} = \dfrac{\mu_{\text{wall}} A_{\text{wall}}}{d_{\perp,\text{P}}} \left(\left(v_{\text{p}} - v_{\text{wall}} \right) \dfrac{-r r_{,\zeta}}{r^2 + r_{,\zeta}^2} + w_{\text{wall}} \dfrac{r^2}{r^2 + r_{,\zeta}^2} \right)_{\text{wall}}
\end{cases}
\tag{1.217}
$$

When discretize CVs at wall using Eq. (1.121) and (1.122), the associated a^{m} parameters and viscous source terms should be calculated by Eq. (1.217). And note that

$$
A_{\text{wall}} = A_{\text{s}}^{\text{tr}} \quad or \quad A_{\text{n}}^{\text{tr}} \tag{1.218}
$$

The non-slip wall boundary condition will make the a^{m} parameters to be different for each momentum equation. This, as discussed previously, is not desirable. Ferziger and Perić[14] pointed out that this situation could be resolved by using deferred correction. However there is another possible way. The idea is that find out the common shared potion of the a^{m} parameters and store them in a safe place (I mean a data structure in your computing code). Store the specific portion of a^{m} parameters of each momentum equation. At the time solving any one of the momentum equations, combine the common and specific portion on the fly to obtain the final coefficient matrix for the linear solver.

9.2 Pressure

Use Eq. (1.219) to calculate the pressure on boundaries.

$$
p_{\text{qbn}} = p_{\text{P}} + \nabla p_{\text{P}}^{(n)} \cdot \mathbf{r}_{\text{Pqbn}} \tag{1.219}
$$

where $\mathbf{r}_{\text{Pqbn}}$ is the vector starting from P and pointing to face q. Note that $\mathbf{r}_{\text{Pqbn}}$ should represent a spatial curve like $\mathbf{s}_{\text{PQ}}$ in Section 8.

10 Residual

The normalized residual is utilized in the current article. For a scalar ϕ, if its discretized equation is Eq. (1.122). Then at iteration step n, the normalized residual is

$$Rs_{\mathrm{P}}^{(n)} = \left(\frac{\left| a_{\mathrm{P}}^{\mathrm{m}} \phi_{\mathrm{P}} - \left(\sum_{\mathrm{q,Q}} a_{\mathrm{Q}}^{\mathrm{m}} \phi_{\mathrm{Q}} + S_{\phi} \right) \right|}{\max_{\mathrm{all\ P}} \left(\left| a_{\mathrm{P}}^{\mathrm{m}} \phi_{\mathrm{P}} \right| \right)} \right)^{(n)} \tag{1.220}$$

Eq. (1.220) should be evaluated at every CV center P. Let $Rs_{\phi}^{\mathrm{limit}}$ be the residual limit. A convergence is achieved if

$$\max_{\mathrm{all\ P}} \left(Rs_{\mathrm{P}}^{(n)} \right) \le Rs_{\phi}^{\mathrm{limit}} \tag{1.221}$$

When all the equations satisfy Eq. (1.221), then the over all calculation is converged.

11 Flowchart

The solution procedures of the FVM are illustrated by Fig. 7.

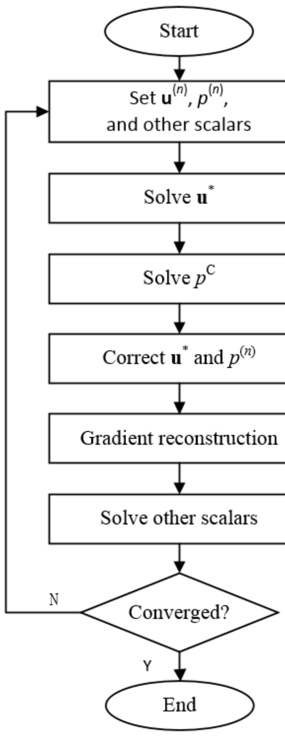

Fig. 7 Flowchart.

12 References

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