Plane in the pixel-disparity space

Yaoyu Hu

Let $\mathbf{p} = [x^{\mathrm{p}}, y^{\mathrm{p}}, d]^{\mathrm{T}}$ be a point in the pixel-disparity space. If $\mathbf{p}$ locate on a plane, then we have

$$(\mathbf{p} - \mathbf{p}_i)^{\mathrm{T}} \mathbf{n}_i = 0 \tag{1}$$

where $\mathbf{p}_i$ and $\mathbf{n}_i$ are the constant vectors associated with the plane in the pixel-disparity space.

$$\mathbf{p}_i = [x_i^{\mathrm{p}}, y_i^{\mathrm{p}}, d_i]^{\mathrm{T}} \tag{2}$$

$$\mathbf{n}_i = [n_{xi}, n_{yi}, n_{di}]^{\mathrm{T}} \tag{3}$$

From the pin-hole camera model, we have the following relationships between the coordinates defined in the 3D camera frame, $\mathbf{p}^{\mathrm{c}} = [x^{\mathrm{c}}, y^{\mathrm{c}}, z^{\mathrm{c}}]$, and the pixel-disparity space, $\mathbf{p}$.

$$x^{\mathrm{p}} = \frac{f}{z^{\mathrm{c}}} x^{\mathrm{c}} + c_x \tag{4}$$

$$y^{\mathrm{p}} = \frac{f}{z^{\mathrm{c}}} y^{\mathrm{c}} + c_y \tag{5}$$

$$d = \frac{bf}{z^{\mathrm{c}}} \tag{6}$$

f and b are the focal length and baseline, respectively. c_x and c_y are the principal point pixel coordinates. f, b, c_x, and c_y are constants. Eq. 1 can be re-written as

$$[x^{\mathrm{p}} - x_i^{\mathrm{p}}, y^{\mathrm{p}} - y_i^{\mathrm{p}}, d - d_i] [n_{xi}, n_{yi}, n_{di}]^{\mathrm{T}} = 0 \tag{7}$$

Insert Eq. 4, 5, and 6 in to Eq. 7 and expand the equation, then we get.

$$\left(\frac{f}{z^{\mathrm{c}}} x^{\mathrm{c}} + c_x - x_i^{\mathrm{p}} \right) n_{xi} + \left(\frac{f}{z^{\mathrm{c}}} y^{\mathrm{c}} + c_y - y_i^{\mathrm{p}} \right) n_{yi} + \left(\frac{bf}{z^{\mathrm{c}}} - d_i \right) n_{di} = 0 \tag{8}$$

Here, we assume that $z^{\mathrm{c}} \neq 0$. Multiply $\frac{z^{\mathrm{c}}}{f}$ on both sides of Eq. 8 and rearrange the terms

$$n_{xi} x^{\mathrm{c}} + n_{yi} y^{\mathrm{c}} + \frac{1}{f} \left((c_x - x_i^{\mathrm{p}}) n_{xi} + (c_y - y_i^{\mathrm{p}}) n_{yi} - d_i n_{di} \right) z^{\mathrm{c}} + b n_{di} = 0$$

that is

$$A^{\mathrm{c}} x^{\mathrm{c}} + B^{\mathrm{c}} y^{\mathrm{c}} + C^{\mathrm{c}} z^{\mathrm{c}} + D^{\mathrm{c}} = 0 \tag{9}$$

where

$$A^{\mathrm{c}} = n_{xi} \tag{10}$$

$$B^{\mathrm{c}} = n_{yi} \tag{11}$$

$$C^{\mathrm{c}} = \frac{1}{f} \left((c_x - x_i^{\mathrm{p}}) n_{xi} + (c_y - y_i^{\mathrm{p}}) n_{yi} - d_i n_{di} \right) \tag{12}$$

$$D^{\mathrm{c}} = b n_{di} \tag{13}$$

Because A^{c}, B^{c}, C^{c}, and D^{c} are all constants, Eq. 9 shows that the 3D point, which is associated with $\mathbf{p}$ on a plane in the pixel-disparity space and is observed in the 3D camera frame with coordinate $\mathbf{p}^{\mathrm{c}}$, is also lying on a spatial plane in the 3D camera frame.

Yaoyu Hu is a post-doc at Carnegie Mellon University. yaoyuh@andrew.cmu.edu