Solution to Ex. 6.29

of *Turbulent Flows* by Stephen B. Pope, 2000

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Show that in isotropic turbulence,

$$E(\kappa) = -\kappa \frac{\mathrm{d}}{\mathrm{d}\kappa}\left(\frac{1}{2}E_{ii}(\kappa)\right) = -\kappa \frac{\mathrm{d}}{\mathrm{d}\kappa}\left(\frac{1}{2}E_{11}(\kappa) + E_{22}(\kappa)\right) \tag{1}$$

Solution

Similar to Eq. (6.216)

$$\begin{aligned} E_{22}(\kappa_1) &= 2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{E(\kappa)}{4\pi\kappa^2}\left(1-\frac{\kappa_2^2}{\kappa^2}\right)\mathrm{d}\kappa_2\mathrm{d}\kappa_3 \\ &= 2\int_{-\infty}^{+\infty} \frac{E(\kappa)}{4\pi\kappa^2}\left(1-\frac{\kappa_2^2}{\kappa^2}\right)2\pi\kappa\mathrm{d}\kappa \\ &= \int_{\kappa_1}^{+\infty} \frac{E(\kappa)}{\kappa}\left(1-\frac{\kappa_2^2}{\kappa^2}\right)\mathrm{d}\kappa \end{aligned} \tag{2}$$

Let $\kappa_1 = \kappa$,

$$E_{22}(\kappa) = \int_{\kappa}^{+\infty} \frac{E(\kappa)}{\kappa}\left(1-\frac{\kappa_2^2}{\kappa^2}\right)\mathrm{d}\kappa \tag{3}$$

Calculate the derivative with respect to κ

$$\frac{\mathrm{d}}{\mathrm{d}\kappa}E_{22}(\kappa) = \int_{\kappa}^{+\infty} \frac{E(\kappa')}{\kappa'}\left(1-\frac{\kappa_2^2}{\kappa'^2}\right)\mathrm{d}\kappa' = -\frac{E(\kappa)}{\kappa}\left(1-\frac{\kappa_2^2}{\kappa^2}\right) \tag{4}$$

Multiply κ on both sides.

$$-\kappa\frac{\mathrm{d}}{\mathrm{d}\kappa}E_{22}(\kappa) = E(\kappa)\left(1-\frac{\kappa_2^2}{\kappa^2}\right) \tag{5}$$

Similarly

$$-\kappa\frac{\mathrm{d}}{\mathrm{d}\kappa}E_{11}(\kappa)=E(\kappa)\left(1-\frac{\kappa_1^2}{\kappa^2}\right) \tag{6}$$

With the fact that

$$E_{33}(\kappa)=E_{22}(\kappa) \tag{7}$$

we have

$$\kappa_2^2=\kappa_3^2 \tag{8}$$

$$-\kappa\frac{\mathrm{d}}{\mathrm{d}\kappa}E_{22}(\kappa)=E(\kappa)\left(1-\frac{\frac{1}{2}\left(\kappa^2-\kappa_1^2\right)}{\kappa^2}\right)=\frac{1}{2}E(\kappa)\left(1+\frac{\kappa_1^2}{\kappa^2}\right) \tag{9}$$

Write 1/2Eq. (6) plus Eq. (9)

$$-\kappa\frac{\mathrm{d}}{\mathrm{d}\kappa}\frac{1}{2}E_{11}(\kappa)-\kappa\frac{\mathrm{d}}{\mathrm{d}\kappa}E_{22}(\kappa)=\frac{1}{2}E(\kappa)\left(1-\frac{\kappa_1^2}{\kappa^2}\right)+\frac{1}{2}E(\kappa)\left(1+\frac{\kappa_1^2}{\kappa^2}\right)$$

$$\Rightarrow$$

$$-\kappa\frac{\mathrm{d}}{\mathrm{d}\kappa}\left(\frac{1}{2}E_{11}(\kappa)+E_{22}(\kappa)\right)=E(\kappa) \tag{10}$$

$$\Rightarrow$$

$$\kappa\frac{\mathrm{d}}{\mathrm{d}\kappa}\left(\frac{1}{2}E_{ii}(\kappa)\right)=E(\kappa)$$