Solution to Ex. 6.28

of *Turbulent Flows* by Stephen B. Pope, 2000

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April 5th, 2017

Differentiate Eq. (6.216) to obtain

$$\frac{\mathrm{d}E_{11}(\kappa_1)}{\mathrm{d}\kappa_1} = -2\kappa_1 \int_{\kappa_1}^{+\infty} E(\kappa)\kappa^{-3}\mathrm{d}\kappa \tag{1}$$

$$\frac{\mathrm{d}^2 E_{11}(\kappa_1)}{d\kappa_1^2} = \frac{2E(\kappa_1)}{\kappa_1^2} - 2\int_{\kappa_1}^{+\infty} E(\kappa)\kappa^{-3}\mathrm{d}\kappa \tag{2}$$

Hence verify Eq. (6.127). Use Eq. (6.221) to show that $E_{11}(\kappa_1)$ is a monotonically decreasing function of κ_1, and hence is maximum at $\kappa_1 = 0$.

Solution

Using the rule of "differentiation under the integral"[1], we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\kappa_1}E_{11}(\kappa_1) &= \frac{\mathrm{d}}{\mathrm{d}\kappa_1}\int_{\kappa_1}^{+\infty}\frac{E(\kappa)}{\kappa}\left(1-\frac{\kappa_1^2}{\kappa^2}\right)\mathrm{d}\kappa \\
&= \underbrace{-\frac{E(\kappa_1)}{\kappa_1}\left(1-\frac{\kappa_1^2}{\kappa_1^2}\right)}_{0} + \int_{\kappa_1}^{+\infty}\frac{\mathrm{d}}{\mathrm{d}\kappa_1}\frac{E(\kappa)}{\kappa}\left(1-\frac{\kappa_1^2}{\kappa^2}\right)\mathrm{d}\kappa \\
&= -2\kappa_1\int_{\kappa_1}^{+\infty}\frac{E(\kappa)}{\kappa^3}\mathrm{d}\kappa
\end{aligned} \tag{3}$$

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}\kappa_1^2}E_{11}(\kappa_1) &= \frac{\mathrm{d}}{\mathrm{d}\kappa_1}\frac{\mathrm{d}}{\mathrm{d}\kappa_1}E_{11}(\kappa_1) \\
&= \frac{\mathrm{d}}{\mathrm{d}\kappa_1}\left(-2\kappa_1\int_{\kappa_1}^{+\infty}\frac{E(\kappa)}{\kappa^3}\mathrm{d}\kappa\right) \\
&= -2\kappa_1\left(-\frac{E(\kappa_1)}{\kappa_1^3} + \int_{\kappa_1}^{+\infty}\underbrace{\frac{\mathrm{d}}{\mathrm{d}\kappa_1}\frac{E(\kappa)}{\kappa^3}}_{=0}\mathrm{d}\kappa\right) - 2\int_{\kappa_1}^{+\infty}\frac{E(\kappa)}{\kappa^3}\mathrm{d}\kappa \\
&= \frac{2E(\kappa_1)}{\kappa_1^2} - 2\int_{\kappa_1}^{+\infty}\frac{E(\kappa)}{\kappa^3}\mathrm{d}\kappa
\end{aligned} \tag{4}$$

Rearrange Eq. (3)

$$\frac{1}{\kappa_1}\frac{\mathrm{d}}{\mathrm{d}\kappa_1}E_{11}\left(\kappa_1\right)=-2\int_{\kappa_1}^{+\infty}\frac{E\left(\kappa\right)}{\kappa^3}\mathrm{d}\kappa \tag{5}$$

Insert Eq. (5) into Eq. (4)

$$\frac{\mathrm{d}^2}{\mathrm{d}\kappa_1^2}E_{11}\left(\kappa_1\right)=\frac{2E\left(\kappa_1\right)}{\kappa_1^2}+\frac{1}{\kappa_1}\frac{\mathrm{d}}{\mathrm{d}\kappa_1}E_{11}\left(\kappa_1\right) \tag{6}$$

Again, rearrange the terms

$$\begin{aligned}\frac{2E\left(\kappa_1\right)}{\kappa_1^3}&=\frac{1}{\kappa_1}\frac{\mathrm{d}^2}{\mathrm{d}\kappa_1^2}E_{11}\left(\kappa_1\right)-\frac{1}{\kappa_1^2}\frac{\mathrm{d}}{\mathrm{d}\kappa_1}E_{11}\left(\kappa_1\right)\\&=\frac{\mathrm{d}}{\mathrm{d}\kappa_1}\left(\frac{1}{\kappa_1}\frac{\mathrm{d}}{\mathrm{d}\kappa_1}E_{11}\left(\kappa_1\right)\right)\end{aligned} \tag{7}$$

Then it is straight forward that

$$E\left(\kappa_1\right)=\frac{1}{2}\kappa_1^3\frac{\mathrm{d}}{\mathrm{d}\kappa_1}\left(\frac{1}{\kappa_1}\frac{\mathrm{d}}{\mathrm{d}\kappa_1}E_{11}\left(\kappa_1\right)\right) \tag{8}$$

Since κ_1 should be non-negative, then Eq. (3) indicates that the first order derivative of $E_{11}\left(\kappa_1\right)$ respect to κ_1 is always negative. Thus $E_{11}\left(\kappa_1\right)$ is a monotonically decreasing function of κ_1 with it maximum value at $\kappa_1 = 0$.

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[1] http://math.stackexchange.com/questions/1743872/derivative-of-an-integral-function/1743902