Solution to Ex. 6.27

of *Turbulent Flows* by Stephen B. Pope, 2000

Yaoyu Hu
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From Eq. (6.180) show that

$$E_{22}\left(\mathbf{e}_1 r_1\right) = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi_{22}\left(\boldsymbol{\kappa}\right)\mathrm{d}\kappa_2\mathrm{d}\kappa_3\right)e^{i\kappa_1 r_1}\mathrm{d}\kappa_1 \tag{1}$$

and from Eq. (6.208) show that

$$R_{22}\left(\mathbf{e}_1 r_1\right) = \int_{-\infty}^{\infty}\frac{1}{2}E_{22}\left(\kappa_1\right)e^{i\kappa_1 r_1}\mathrm{d}\kappa_1 \tag{2}$$

Hence verify Eq. (6.210).

Solution

Let

$$\mathbf{r} = \mathbf{e}_1 r_1 \tag{3}$$

Then Eq. (6.180) turns into

$$\begin{aligned} R_{22}\left(\mathbf{r}\right) &= R_{22}\left(\mathbf{e}_1 r_1\right) \\ &= \iiint\limits_{-\infty,+\infty}\Phi_{22}\left(\boldsymbol{\kappa}\right)e^{i\kappa_1 r_1}\mathrm{d}\kappa_1\mathrm{d}\kappa_2\mathrm{d}\kappa_3 = \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Phi_{22}\left(\boldsymbol{\kappa}\right)\mathrm{d}\kappa_2\mathrm{d}\kappa_3\right)e^{i\kappa_1 r_1}\mathrm{d}\kappa_1 \end{aligned} \tag{4}$$

From Eq. (6.208)

$$\begin{aligned} R_{22}\left(\mathbf{e}_1 r_1\right) &= \int_0^{+\infty}E_{22}\left(\kappa_1\right)\cos\left(\kappa_1\cdot r_1\right)\mathrm{d}\kappa_1 \\ &= \frac{1}{2}\left[\int_{-\infty}^{+\infty}E_{22}\left(\kappa_1\right)\cos\left(\kappa_1\cdot r_1\right)\mathrm{d}\kappa_1 + i\underbrace{\int_{-\infty}^{+\infty}E_{22}\left(\kappa_1\right)\sin\left(\kappa_1\cdot r_1\right)\mathrm{d}\kappa_1}_{*}\right] \\ &= \frac{1}{2}\int_{-\infty}^{+\infty}E_{22}\left(\kappa_1\right)\left(\cos\left(\kappa_1\cdot r_1\right)+i\sin\left(\kappa_1\cdot r_1\right)\right)\mathrm{d}\kappa_1 \\ &= \int_{-\infty}^{+\infty}\frac{1}{2}E_{22}\left(\kappa_1\right)e^{i\kappa_1\cdot r_1}\mathrm{d}\kappa_1 \end{aligned} \tag{5}$$

where the term marked with * is zero due to the fact that $E_{22}(\kappa)$ is even and sin() is odd.

Note that Eq. (4) equals Eq. (5)

$$\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Phi_{22}\left(\boldsymbol{\kappa}\right)\mathrm{d}\kappa_2\mathrm{d}\kappa_3\right)e^{i\kappa_1 r_1}\mathrm{d}\kappa_1=\int_{-\infty}^{+\infty}\frac{1}{2}E_{22}\left(\kappa_1\right)e^{i\kappa_1\cdot r_1}\mathrm{d}\kappa_1 \tag{6}$$

This leads to

$$E_{22}\left(\kappa_1\right)e^{i\kappa_1\cdot r_1}=2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Phi_{22}\left(\boldsymbol{\kappa}\right)\mathrm{d}\kappa_2\mathrm{d}\kappa_3 \tag{7}$$