Solution to Ex. 6.24

of *Turbulent Flows* by Stephen B. Pope, 2000

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Show that

$$\oint \mathrm{d}S\left(\kappa\right) = 4\pi\kappa^2 \tag{1}$$

$$\oint \kappa_i \kappa_j \mathrm{d}S\left(\kappa\right) = \frac{4}{3}\pi\kappa^4\delta_{ij} \tag{2}$$

(Hint: argue that the integral in Eq. (2) must be isotropic, i.e., a scalar multiple of δ_{ij})

Solution

Since $S(\kappa)$ is the sphere in wavenumber space with radius κ. $\mathrm{d}S(\kappa)$ is the infinitesimal surface element of S. The integral of Eq. (1) must yields the surface area of a sphere with radius κ. That is

$$\oint \mathrm{d}S\left(\kappa\right) = \int_{-\pi}^{\pi}\int_{0}^{\pi}\kappa^2 \sin\left(\theta\right)\mathrm{d}\theta\mathrm{d}\varphi = 4\pi\kappa^2 \tag{3}$$

where θ and φ are the two azimuthal coordinates variable of a spherical coordinate system, in wavenumber space.

Also in the above spherical coordinate system, κ_i could be expressed by

$$\begin{cases} \kappa_1 = \kappa\sin\left(\theta\right)\cos\left(\varphi\right) \\ \kappa_2 = \kappa\sin\left(\theta\right)\sin\left(\varphi\right) \\ \kappa_3 = \kappa\cos\left(\theta\right) \end{cases} \tag{4}$$

Then for $i = 1$ and $j = 2$

$$\begin{aligned} \oint \kappa_i \kappa_j \mathrm{d}S\left(\kappa\right) &= \int_{-\pi}^{\pi}\int_{0}^{\pi}\kappa\sin\left(\theta\right)\cos\left(\varphi\right)\kappa\sin\left(\theta\right)\sin\left(\varphi\right)\kappa^2\sin\left(\theta\right)\mathrm{d}\theta\mathrm{d}\varphi \\ &= \kappa^4\int_{-\pi}^{\pi}\int_{0}^{\pi}\sin^3\left(\theta\right)\cos\left(\varphi\right)\sin\left(\varphi\right)\mathrm{d}\theta\mathrm{d}\varphi \\ &= 0 \end{aligned} \tag{5}$$

It is easy to verify that for $i \neq j$, the integral similar to Eq. (5) equals zero. As for $i = j$, we have

$$\begin{aligned}
\oint \kappa_1 \kappa_1 \mathrm{d}S(\kappa) &= \int_{-\pi}^{\pi}\int_0^{\pi} \kappa \sin(\theta)\cos(\varphi)\kappa \sin(\theta)\cos(\varphi)\kappa^2 \sin(\theta)\,\mathrm{d}\theta \mathrm{d}\varphi \\
&= \kappa^4 \int_{-\pi}^{\pi}\int_0^{\pi} \sin^3(\theta)\cos^2(\varphi)\,\mathrm{d}\theta \mathrm{d}\varphi \\
&= \kappa^4 \int_0^{\pi} \sin^3(\theta)\,\mathrm{d}\theta \int_{-\pi}^{\pi} \cos^2(\varphi)\,\mathrm{d}\varphi \\
&= \kappa^4 \int_0^{\pi} \frac{1}{4}\big(3\sin(\theta) - \sin(3\theta)\big)\mathrm{d}\theta \int_{-\pi}^{\pi} \frac{1}{2}\big(1+\cos(2\varphi)\big)\mathrm{d}\varphi \\
&= \frac{4}{3}\pi\kappa^4
\end{aligned} \tag{6}$$

$$\begin{aligned}
\oint \kappa_2 \kappa_2 \mathrm{d}S(\kappa) &= \int_{-\pi}^{\pi}\int_0^{\pi} \kappa \sin(\theta)\sin(\varphi)\kappa \sin(\theta)\sin(\varphi)\kappa^2 \sin(\theta)\,\mathrm{d}\theta \mathrm{d}\varphi \\
&= \kappa^4 \int_{-\pi}^{\pi}\int_0^{\pi} \sin^3(\theta)\sin^2(\varphi)\,\mathrm{d}\theta \mathrm{d}\varphi \\
&= \kappa^4 \int_0^{\pi} \sin^3(\theta)\,\mathrm{d}\theta \int_{-\pi}^{\pi} \sin^2(\varphi)\,\mathrm{d}\varphi \\
&= \kappa^4 \int_0^{\pi} \frac{1}{4}\big(3\sin(\theta) - \sin(3\theta)\big)\mathrm{d}\theta \int_{-\pi}^{\pi} \frac{1}{2}\big(1-\cos(2\theta)\big)\mathrm{d}\varphi \\
&= \frac{4}{3}\pi\kappa^4
\end{aligned} \tag{7}$$

$$\begin{aligned}
\oint \kappa_3 \kappa_3 \mathrm{d}S(\kappa) &= \int_{-\pi}^{\pi}\int_0^{\pi} \kappa \cos(\theta)\kappa \cos(\theta)\kappa^2 \sin(\theta)\,\mathrm{d}\theta \mathrm{d}\varphi \\
&= \kappa^4 \int_{-\pi}^{\pi}\int_0^{\pi} \cos^2(\theta)\sin(\theta)\,\mathrm{d}\theta \mathrm{d}\varphi \\
&= -\kappa^4 \int_0^{\pi} \cos^2(\theta)\,\mathrm{d}\cos(\theta)\,\mathrm{d}\varphi \int_{-\pi}^{\pi} \mathrm{d}\varphi \\
&= \frac{4}{3}\pi\kappa^4
\end{aligned} \tag{8}$$

Summarize Eq. (5) to Eq. (8) we have

$$\oint \kappa_i \kappa_j \mathrm{d}S(\kappa) = \frac{4}{3}\pi\kappa^4\delta_{ij} \tag{9}$$