Solution to Ex. 6.23 (May not be correct)

of *Turbulent Flows* by Stephen B. Pope, 2000

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April 1st, 2017

Starting from the spectral representation for **u**(**x**) (Eq. (6.119)), show that the spectral representation of $\partial u_i / \partial x_k$ is

$$\frac{\partial u_i}{\partial x_k} = \sum_{\boldsymbol{\kappa}} i\kappa_k \hat{u}_i\left(\boldsymbol{\kappa}\right) e^{i\boldsymbol{\kappa}\cdot\mathbf{x}} \tag{1}$$

Hence show the relations

$$\left\langle \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_l} \right\rangle = \sum_{\boldsymbol{\kappa}} \kappa_k \kappa_l \hat{R}_{ij}\left(\boldsymbol{\kappa}\right) = \iiint\limits_{-\infty,+\infty} \bar{\kappa}_k \bar{\kappa}_l \Phi_{ij}\left(\bar{\boldsymbol{\kappa}}\right) \mathrm{d}\bar{\boldsymbol{\kappa}} \tag{2}$$

$$\varepsilon = \sum_{\boldsymbol{\kappa}} 2\nu\kappa^2 \hat{E}\left(\kappa\right) = \iiint\limits_{-\infty,+\infty} 2\nu\bar{\kappa}^2 \frac{1}{2} \Phi_{ii}\left(\bar{\boldsymbol{\kappa}}\right) \mathrm{d}\bar{\boldsymbol{\kappa}} \tag{3}$$

Solution

It could be obtained easily that

$$\frac{\partial u_i}{\partial u_k} = \frac{\partial}{\partial x_k} \sum_{\boldsymbol{\kappa}} \hat{u}_i\left(\boldsymbol{\kappa}\right) e^{i\boldsymbol{\kappa}\cdot\mathbf{x}} = \sum_{\boldsymbol{\kappa}} \hat{u}_i\left(\boldsymbol{\kappa}\right) \frac{\partial}{\partial x_k} e^{i\boldsymbol{\kappa}\cdot\mathbf{x}} = \sum_{\boldsymbol{\kappa}} i\kappa_k \hat{u}_i\left(\boldsymbol{\kappa}\right) e^{i\boldsymbol{\kappa}\cdot\mathbf{x}} \tag{4}$$

Using Eq. (4), we can write

$$
\begin{aligned}
\left\langle \frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_l}\right\rangle &= \left\langle \sum_{\boldsymbol{\kappa}'} i\kappa'_k \hat{u}_i\left(\boldsymbol{\kappa}'\right)e^{i\boldsymbol{\kappa}'\cdot\mathbf{x}} \sum_{\boldsymbol{\kappa}} i\kappa_l \hat{u}_j\left(\boldsymbol{\kappa}\right)e^{i\boldsymbol{\kappa}\cdot\mathbf{x}} \right\rangle \\
&= \left\langle \sum_{-\boldsymbol{\kappa}'} -i\kappa_k \hat{u}_i\left(-\boldsymbol{\kappa}'\right)e^{-i\boldsymbol{\kappa}'\cdot\mathbf{x}} \sum_{\boldsymbol{\kappa}} i\kappa_l \hat{u}_j\left(\boldsymbol{\kappa}\right)e^{i\boldsymbol{\kappa}\cdot\mathbf{x}} \right\rangle \\
&= \left\langle \sum_{\boldsymbol{\kappa}}\sum_{-\boldsymbol{\kappa}'} \kappa_k\kappa_l e^{-i\boldsymbol{\kappa}'\cdot\mathbf{x}} e^{i\boldsymbol{\kappa}\cdot\mathbf{x}} \hat{u}_i\left(-\boldsymbol{\kappa}'\right)\hat{u}_j\left(\boldsymbol{\kappa}\right)\right\rangle \\
&= \sum_{\boldsymbol{\kappa}}\sum_{-\boldsymbol{\kappa}'} \kappa_k\kappa_l \left\langle e^{-i\boldsymbol{\kappa}'\cdot\mathbf{x}} e^{i\boldsymbol{\kappa}\cdot\mathbf{x}}\right\rangle \left\langle \hat{u}_i\left(-\boldsymbol{\kappa}'\right)\hat{u}_j\left(\boldsymbol{\kappa}\right)\right\rangle \\
&\overset{\text{only at } \boldsymbol{\kappa}'=\boldsymbol{\kappa}}{=} \sum_{\boldsymbol{\kappa}} \kappa_k\kappa_l \left\langle \hat{u}_i\left(-\boldsymbol{\kappa}\right)\hat{u}_j\left(\boldsymbol{\kappa}\right)\right\rangle \\
&= \sum_{\boldsymbol{\kappa}} \kappa_k\kappa_l \hat{R}_{ij}\left(\boldsymbol{\kappa}\right)
\end{aligned}
\tag{5}
$$

The integral (this may be not correct)

$$
\begin{aligned}
\iiint\limits_{-\infty,+\infty} \bar{\kappa}_k\bar{\kappa}_l \Phi_{ij}\left(\bar{\boldsymbol{\kappa}}\right)\mathrm{d}\bar{\boldsymbol{\kappa}} &= \iiint\limits_{-\infty,+\infty} \bar{\kappa}_k\bar{\kappa}_l \sum_{\boldsymbol{\kappa}} \delta\left(\bar{\boldsymbol{\kappa}}-\boldsymbol{\kappa}\right)\hat{R}_{ij}\left(\boldsymbol{\kappa},t\right)\mathrm{d}\bar{\boldsymbol{\kappa}} \\
&= \iiint\limits_{-\infty,+\infty} \sum_{\boldsymbol{\kappa}} \kappa_k\kappa_l \delta\left(\bar{\boldsymbol{\kappa}}-\boldsymbol{\kappa}\right)\hat{R}_{ij}\left(\boldsymbol{\kappa},t\right)\mathrm{d}\bar{\boldsymbol{\kappa}} \\
&= \sum_{\boldsymbol{\kappa}} \left(\kappa_k\kappa_l \hat{R}_{ij}\left(\boldsymbol{\kappa}\right) \iiint\limits_{-\infty,+\infty} \delta\left(\bar{\boldsymbol{\kappa}}-\boldsymbol{\kappa}\right)\mathrm{d}\bar{\boldsymbol{\kappa}} \right) \\
&= \sum_{\boldsymbol{\kappa}} \kappa_k\kappa_l \hat{R}_{ij}\left(\boldsymbol{\kappa}\right) \\
&= \left\langle \frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_l}\right\rangle
\end{aligned}
\tag{6}
$$

For dissipation rate, we have

$$
\begin{aligned}
\varepsilon &= 2\nu\left\langle s_{ij}s_{ij}\right\rangle \\
&= \nu\left\langle \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right\rangle \\
&= \nu\left\langle \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\right\rangle + \nu\left\langle \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right\rangle
\end{aligned}
\tag{7}
$$

Invoking Eq. (5), Eq. (7) is (this may not be correct)

$$
\begin{aligned}
\varepsilon &= \nu \left\langle \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} \right\rangle + \nu \left\langle \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \right\rangle \\
&= \nu \sum_{\boldsymbol{\kappa}} \kappa_j \kappa_j \hat{R}_{ii}\left(\boldsymbol{\kappa}\right) + \nu \sum_{\boldsymbol{\kappa}} \kappa_j \kappa_i \hat{R}_{ij}\left(\boldsymbol{\kappa}\right)
\end{aligned} \tag{8}
$$

Recall that Eq. (6.172) tells us that

$$
\kappa_i \hat{R}_{ij}\left(\boldsymbol{\kappa}\right) = 0 \tag{9}
$$

So Eq. (8) could be expressed as

$$
\varepsilon = \nu \sum_{\boldsymbol{\kappa}} \kappa_j \kappa_j \hat{R}_{ii}\left(\boldsymbol{\kappa}\right) = \sum_{\boldsymbol{\kappa}} 2\nu\kappa^2 \hat{E}\left(\boldsymbol{\kappa}\right) = \iiint_{-\infty,+\infty} 2\nu\bar{\boldsymbol{\kappa}}^2 \frac{1}{2} \Phi_{ii}\left(\bar{\boldsymbol{\kappa}}, t\right) \mathrm{d}\bar{\boldsymbol{\kappa}} \tag{10}
$$