Solution to Ex. 6.21

of *Turbulent Flows* by Stephen B. Pope, 2000

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Let **Y** be any constant vector, and define $\hat{g}(\boldsymbol{\kappa}) = \mathbf{Y} \cdot \hat{\mathbf{u}}(\boldsymbol{\kappa})$. Obtain the result

$$Y_i Y_j \hat{R}_{ij}(\boldsymbol{\kappa}) = \left\langle \hat{g}^*(\boldsymbol{\kappa}) \hat{g}(\boldsymbol{\kappa}) \right\rangle \geq 0 \tag{1}$$

to show that both $\hat{R}_{ij}(\boldsymbol{\kappa})$ and $\Phi_{ij}(\bar{\boldsymbol{\kappa}})$ are positive semi-definite, i.e.,

$$Y_i Y_j \hat{R}_{ij}(\boldsymbol{\kappa}) \geq 0 \tag{2}$$

$$Y_i Y_j \Phi_{ij}(\bar{\boldsymbol{\kappa}}) \geq 0 \tag{3}$$

for all **Y**. (This is a stronger result than Eq. (6.169).)

Solution

I think **Y** should be a vector with real components. Then

$$Y_i Y_j \hat{R}_{ij}(\boldsymbol{\kappa}) = \left\langle Y_i \hat{u}_i^*(\boldsymbol{\kappa}, t) Y_j \hat{u}_j(\boldsymbol{\kappa}, t) \right\rangle = \left\langle \mathbf{Y} \cdot \hat{\mathbf{u}}^* \times \mathbf{Y} \cdot \hat{\mathbf{u}} \right\rangle = \left\langle \hat{g}^*(\boldsymbol{\kappa}) \hat{g}(\boldsymbol{\kappa}) \right\rangle \geq 0 \tag{4}$$

This means that $\hat{R}_{ij}(\boldsymbol{\kappa})$ is positive definite.

The same holds with $\Phi_{ij}(\boldsymbol{\kappa})$ from its definition (Eq. (6.155)).