# Solution to Ex. 6.20

of *Turbulent Flows* by Stephen B. Pope, 2000

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April 1st, 2017

From the definition of $\hat{R}_{ij}(\boldsymbol{\kappa})$ (Eq. (6.152)) show that

$$\hat{R}_{ij}(\boldsymbol{\kappa}) \geq 0, \text{ for } i = j \tag{1}$$

$$\hat{R}_{ij}(\boldsymbol{\kappa}) \geq 0 \tag{2}$$

From conjugate symmetry show that

$$\hat{R}_{ij}(\boldsymbol{\kappa}) = \hat{R}_{ij}(-\boldsymbol{\kappa}) = \hat{R}_{ij}^*(\boldsymbol{\kappa}) \tag{3}$$

From the incompressibility condition $\boldsymbol{\kappa} \cdot \hat{\mathbf{u}}(\boldsymbol{\kappa}) = 0$, show that

$$\kappa_i \hat{R}_{ij}(\boldsymbol{\kappa}) = \kappa_j \hat{R}_{ij}(\boldsymbol{\kappa}) = 0 \tag{4}$$

Note that all of these properties also apply to the velocity-spectrum tensor $\Phi_{ij}(\boldsymbol{\kappa})$.

**Solution**

Setting $i = j$, and from the definition of $\hat{R}_{ij}(\boldsymbol{\kappa})$ we could write

$$\hat{R}_{ij}(\boldsymbol{\kappa}, t) \overset{i=j}{=} \left\langle \hat{u}_i^*(\boldsymbol{\kappa}, t) \hat{u}_{j=i}(\boldsymbol{\kappa}, t) \right\rangle = \left\langle \left\| \hat{u}_i \right\|^2 \right\rangle \geq 0 \tag{5}$$

where $\|z\|$ is the operation that calculates the modulus of a complex number $z$. So it is quite straight forward that

$$\hat{R}_{ii}(\boldsymbol{\kappa}, t) = \left\langle \left\| \hat{u}_1 \right\| \right\rangle + \left\langle \left\| \hat{u}_2 \right\| \right\rangle + \left\langle \left\| \hat{u}_3 \right\| \right\rangle \geq 0 \tag{6}$$

Again from the definition of $\hat{R}_{ij}(\boldsymbol{\kappa}, t)$

$$\hat{R}_{ij}(\boldsymbol{\kappa}, t) = \left\langle \hat{u}_i^*(\boldsymbol{\kappa}, t) \hat{u}_j(\boldsymbol{\kappa}, t) \right\rangle = \left\langle \hat{u}_i(-\boldsymbol{\kappa}, t) \hat{u}_j^*(-\boldsymbol{\kappa}, t) \right\rangle = \hat{R}_{ji}(-\boldsymbol{\kappa}, t) \tag{7}$$

For $\hat{R}^*_{ji}(\boldsymbol{\kappa},t)$

$$\hat{R}^*_{ji}(\boldsymbol{\kappa},t)=\left(\left\langle \hat{u}_i(\boldsymbol{\kappa},t)\hat{u}^*_j(\boldsymbol{\kappa},t)\right\rangle\right)^*=\left\langle \hat{u}^*_i(\boldsymbol{\kappa},t)\hat{u}_j(\boldsymbol{\kappa},t)\right\rangle=\hat{R}_{ij}(\boldsymbol{\kappa},t) \quad (8)$$

Thus Eq. (3) holds.

Let's examine Eq. (4)

$$\begin{aligned}\kappa_i\hat{R}_{ij}(\boldsymbol{\kappa}) &= \kappa_i\left\langle \hat{u}^*_i(\boldsymbol{\kappa},t)\hat{u}_j(\boldsymbol{\kappa},t)\right\rangle \\ &= -\left\langle -\kappa_i\hat{u}^*_i(\boldsymbol{\kappa},t)\hat{u}_j(\boldsymbol{\kappa},t)\right\rangle \\ &= -\left\langle -\kappa_i\hat{u}_i(-\boldsymbol{\kappa},t)\hat{u}_j(\boldsymbol{\kappa},t)\right\rangle \\ &= -\left\langle (-\boldsymbol{\kappa})\cdot\hat{\mathbf{u}}(-\boldsymbol{\kappa},t)\hat{u}_j(\boldsymbol{\kappa},t)\right\rangle \\ &= 0\end{aligned} \quad (9)$$

And

$$\kappa_j\hat{R}_{ij}(\boldsymbol{\kappa})=\kappa_j\left\langle \hat{u}^*_i(\boldsymbol{\kappa},t)\hat{u}_j(\boldsymbol{\kappa},t)\right\rangle=\left\langle \hat{u}^*_i(\boldsymbol{\kappa},t)\kappa_j\hat{u}_j(\boldsymbol{\kappa},t)\right\rangle=0 \quad (10)$$