Solution to Ex. 6.19

of *Turbulent Flows* by Stephen B. Pope, 2000

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Through the substitutions

$$u_i(\mathbf{x}) = \sum_{\boldsymbol{\kappa}'} e^{i\boldsymbol{\kappa}'\cdot\mathbf{x}} \hat{u}_i(\boldsymbol{\kappa}') = \sum_{\boldsymbol{\kappa}'} e^{-i\boldsymbol{\kappa}'\cdot\mathbf{x}} \hat{u}_i^*(\boldsymbol{\kappa}') \tag{1}$$

$$u_j(\mathbf{x}+\mathbf{r}) = \sum_{\boldsymbol{\kappa}} e^{i\boldsymbol{\kappa}\cdot(\mathbf{x}+\mathbf{r})} \hat{u}_j(\boldsymbol{\kappa}) \tag{2}$$

show that, in homogeneous turbulence,

$$R_{ij}(\mathbf{r}) = \left\langle R_{ij}(\mathbf{r}) \right\rangle_L = \sum_{\boldsymbol{\kappa}} e^{i\boldsymbol{\kappa}\cdot\mathbf{r}} \left\langle \hat{u}_i^*(\boldsymbol{\kappa}) \hat{u}_j(\boldsymbol{\kappa}) \right\rangle \tag{3}$$

hence establishing Eq. (6.154).

Solution

The tow-point velocity correlation is

$$\begin{aligned}
R_{ij}(\mathbf{r}) &= \left\langle u_i(\mathbf{x}) u_j(\mathbf{x}+\mathbf{r}) \right\rangle \\
&= \left\langle \left\langle u_i(\mathbf{x}) u_j(\mathbf{x}+\mathbf{r}) \right\rangle \right\rangle_L \\
&= \left\langle \left\langle \sum_{\boldsymbol{\kappa}'} e^{-i\boldsymbol{\kappa}'\cdot\mathbf{x}} \hat{u}_i^*(\boldsymbol{\kappa}') \sum_{\boldsymbol{\kappa}} e^{i\boldsymbol{\kappa}\cdot(\mathbf{x}+\mathbf{r})} \hat{u}_j(\boldsymbol{\kappa}) \right\rangle \right\rangle_L \\
&= \left\langle \left\langle \sum_{\boldsymbol{\kappa}} \sum_{\boldsymbol{\kappa}'} e^{-i\boldsymbol{\kappa}'\cdot\mathbf{x}} e^{i\boldsymbol{\kappa}\cdot(\mathbf{x}+\mathbf{r})} \hat{u}_i^*(\boldsymbol{\kappa}') \hat{u}_j(\boldsymbol{\kappa}) \right\rangle \right\rangle_L \\
&= \left\langle \sum_{\boldsymbol{\kappa}} \left\langle \sum_{\boldsymbol{\kappa}'} e^{i(\boldsymbol{\kappa}-\boldsymbol{\kappa}')\cdot\mathbf{x}} \hat{u}_i^*(\boldsymbol{\kappa}') \hat{u}_j(\boldsymbol{\kappa}) \right\rangle_L e^{i\boldsymbol{\kappa}\cdot\mathbf{r}} \right\rangle \\
&= \left\langle \sum_{\boldsymbol{\kappa}} \hat{u}_i^*(\boldsymbol{\kappa}) \hat{u}_j(\boldsymbol{\kappa}) e^{i\boldsymbol{\kappa}\cdot\mathbf{r}} \right\rangle \\
&= \sum_{\boldsymbol{\kappa}} \left\langle \hat{u}_i^*(\boldsymbol{\kappa}) \hat{u}_j(\boldsymbol{\kappa}) \right\rangle e^{i\boldsymbol{\kappa}\cdot\mathbf{r}} \\
&= \sum_{\boldsymbol{\kappa}} \hat{R}_{ij}(\boldsymbol{\kappa}) e^{i\boldsymbol{\kappa}\cdot\mathbf{r}}
\end{aligned} \tag{4}$$

Eq. (6.154) holds.