Solution to Ex. 6.18

of *Turbulent Flows* by Stephen B. Pope, 2000

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Show that the covariance of two Fourier coefficients of velocity can be expressed as

$$\begin{aligned}\left\langle \hat{u}_i\left(\boldsymbol{\kappa}',t\right)\hat{u}_j\left(\boldsymbol{\kappa},t\right)\right\rangle &= \left\langle F_{\boldsymbol{\kappa}}\left\{u_i\left(\mathbf{x}',t\right)\right\}F_{\boldsymbol{\kappa}}\left\{u_j\left(\mathbf{x},t\right)\right\}\right\rangle \\ &= \left\langle \left\langle u_i\left(\mathbf{x}',t\right)e^{-i\boldsymbol{\kappa}'\cdot\mathbf{x}'}\right\rangle_L \left\langle u_j\left(\mathbf{x},t\right)e^{-i\boldsymbol{\kappa}\cdot\mathbf{x}}\right\rangle_L\right\rangle \\ &= \frac{1}{L^6}\int_0^L\cdots\int_0^L \left\langle u_i\left(\mathbf{x}',t\right)u_j\left(\mathbf{x},t\right)\right\rangle e^{-i\left(\boldsymbol{\kappa}'\cdot\mathbf{x}'+\boldsymbol{\kappa}\cdot\mathbf{x}\right)}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{x}'\end{aligned} \tag{1}$$

With the substitution **x** = **x′** + **r**, and from the fact that in homogenous turbulence the two-point correlation $R_{ij}(\mathbf{r},t)$ is independent of position, show that the last result can be re-expressed as

$$\begin{aligned}\left\langle \hat{u}_i\left(\boldsymbol{\kappa}',t\right)\hat{u}_j\left(\boldsymbol{\kappa},t\right)\right\rangle &= \left\langle R_{ij}\left(\mathbf{r},t\right)e^{-i\boldsymbol{\kappa}\cdot\mathbf{r}}\right\rangle_L \left\langle e^{-i\mathbf{x}\cdot\left(\boldsymbol{\kappa}'+\boldsymbol{\kappa}\right)}\right\rangle_L \\ &= F_{\boldsymbol{\kappa}}\left\{R_{ij}\left(\mathbf{r},t\right)\right\}\delta_{\boldsymbol{\kappa}',-\boldsymbol{\kappa}'}\end{aligned} \tag{2}$$

(Hint: see Eq. (E.22).) Hence, by setting **κ**’ = -**κ**, verify Eq. (6.153).

Solution

From Eq. (6.116), Eq. (1) could be written

$$\begin{aligned}\left\langle \hat{u}_i\left(\boldsymbol{\kappa}',t\right)\hat{u}_j\left(\boldsymbol{\kappa},t\right)\right\rangle &= \left\langle F_{\boldsymbol{\kappa}}\left\{u_i\left(\mathbf{x}',t\right)\right\}F_{\boldsymbol{\kappa}}\left\{u_j\left(\mathbf{x},t\right)\right\}\right\rangle \\ &= \left\langle \left\langle u_i\left(\mathbf{x}',t\right)e^{-i\boldsymbol{\kappa}'\cdot\mathbf{x}'}\right\rangle_L \left\langle u_j\left(\mathbf{x},t\right)e^{-i\boldsymbol{\kappa}\cdot\mathbf{x}}\right\rangle_L\right\rangle \\ &= \underbrace{\left\langle \frac{1}{L^3}\int_0^L\int_0^L\int_0^L u_i\left(\mathbf{x}',t\right)e^{-i\boldsymbol{\kappa}'\cdot\mathbf{x}'}\mathrm{d}\mathbf{x}' \frac{1}{L^3}\int_0^L\int_0^L\int_0^L u_j\left(\mathbf{x},t\right)e^{-i\boldsymbol{\kappa}\cdot\mathbf{x}}\mathrm{d}\mathbf{x}\right\rangle}_{*} \\ &= \left\langle \frac{1}{L^6}\int_0^L\cdots\int_0^L u_i\left(\mathbf{x}',t\right)u_j\left(\mathbf{x},t\right)e^{-i\left(\boldsymbol{\kappa}'\cdot\mathbf{x}'+\boldsymbol{\kappa}\cdot\mathbf{x}\right)}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{x}'\right\rangle \\ &= \frac{1}{L^6}\int_0^L\cdots\int_0^L \left\langle u_i\left(\mathbf{x}',t\right)u_j\left(\mathbf{x},t\right)\right\rangle e^{-i\left(\boldsymbol{\kappa}'\cdot\mathbf{x}'+\boldsymbol{\kappa}\cdot\mathbf{x}\right)}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{x}'\end{aligned} \tag{3}$$

We can rewrite Eq. (3) into the form like the * term of Eq. (3)

$$
\begin{aligned}
&\left\langle \hat{u}_i\left(\boldsymbol{\kappa}',t\right)\hat{u}_j\left(\boldsymbol{\kappa},t\right)\right\rangle \\
&=\frac{1}{L^6}\int_0^L\cdots\int_0^L\left\langle u_i\left(\mathbf{x}',t\right)u_j\left(\mathbf{x},t\right)\right\rangle e^{-i\left(\boldsymbol{\kappa}'\cdot\mathbf{x}'+\boldsymbol{\kappa}\cdot\mathbf{x}\right)}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{x}' \\
&=\frac{1}{L^3}\int_0^L\int_0^L\int_0^L\left(\frac{1}{L^3}\int_{-x'}^{L-x'}\int_{-x'}^{L-x'}\int_{-x'}^{L-x'}\left\langle u_i\left(\mathbf{x}',t\right)u_j\left(\mathbf{x}'+\mathbf{r},t\right)\right\rangle e^{-i\left(\boldsymbol{\kappa}'\cdot\mathbf{x}'+\boldsymbol{\kappa}\cdot\mathbf{x}'+\boldsymbol{\kappa}\cdot\mathbf{r}\right)}\mathrm{d}\mathbf{r}\right)\mathrm{d}\mathbf{x}' \\
&=\frac{1}{L^3}\int_0^L\int_0^L\int_0^L\left(\frac{1}{L^3}\int_{-x'}^{L-x'}\int_{-x'}^{L-x'}\int_{-x'}^{L-x'}R_{ij}\left(\mathbf{r},t\right)e^{-i\boldsymbol{\kappa}\cdot\mathbf{r}}\mathrm{d}\mathbf{r}\right)e^{-i\left(\boldsymbol{\kappa}'+\boldsymbol{\kappa}\right)\cdot\mathbf{x}'}\mathrm{d}\mathbf{x}' \\
&=\frac{1}{L^3}\int_0^L\int_0^L\int_0^L F_{\boldsymbol{\kappa}}\left\{R_{ij}\left(\mathbf{r},t\right)\right\}e^{-i\left(\boldsymbol{\kappa}'+\boldsymbol{\kappa}\right)\cdot\mathbf{x}'}\mathrm{d}\mathbf{x}' \\
&=F_{\boldsymbol{\kappa}}\left\{R_{ij}\left(\mathbf{r},t\right)\right\}\frac{1}{L^3}\int_0^L\int_0^L\int_0^L e^{-i\boldsymbol{\kappa}\cdot\mathbf{x}'}e^{i\left(-\boldsymbol{\kappa}'\cdot\right)\mathbf{x}'}\mathrm{d}\mathbf{x}' \\
&=F_{\boldsymbol{\kappa}}\left\{R_{ij}\left(\mathbf{r},t\right)\right\}\delta_{\boldsymbol{\kappa},-\boldsymbol{\kappa}'}
\end{aligned}
\tag{4}
$$

Setting **κ′** = -**κ** in Eq. *equation reference goes here*, we can obtain

$$
\begin{aligned}
\left\langle \hat{u}_i\left(-\boldsymbol{\kappa},t\right)\hat{u}_j\left(\boldsymbol{\kappa},t\right)\right\rangle &=\left\langle \hat{u}_i^*\left(\boldsymbol{\kappa},t\right)\hat{u}_j\left(\boldsymbol{\kappa},t\right)\right\rangle \\
&=\hat{R}_{ij}\left(\boldsymbol{\kappa},t\right) \\
&=F_{\boldsymbol{\kappa}}\left\{R_{ij}\left(\mathbf{r},t\right)\right\}\delta_{\boldsymbol{\kappa},\boldsymbol{\kappa}} \\
&=F_{\boldsymbol{\kappa}}\left\{R_{ij}\left(\mathbf{r},t\right)\right\}
\end{aligned}
\tag{5}
$$