Solution to Ex. 6.17 (Unfinished)

of *Turbulent Flows* by Stephen B. Pope, 2000

Yaoyu Hu
March 31, 2017

Let $\boldsymbol{\kappa}^{\mathrm{a}}$, $\boldsymbol{\kappa}^{\mathrm{b}}$, and $\boldsymbol{\kappa}^{\mathrm{c}}$ be three wavenumber vectors such that

$$\boldsymbol{\kappa}^{\mathrm{a}} + \boldsymbol{\kappa}^{\mathrm{b}} + \boldsymbol{\kappa}^{\mathrm{c}} = 0 \tag{1}$$

and define $\mathbf{a}(t) = \hat{\mathbf{u}}(\boldsymbol{\kappa}^{\mathrm{a}}, t)$, $\mathbf{b}(t) = \hat{\mathbf{u}}(\boldsymbol{\kappa}^{\mathrm{b}}, t)$, and $\mathbf{c}(t) = \hat{\mathbf{u}}(\boldsymbol{\kappa}^{\mathrm{c}}, t)$. Consider a periodic velocity field, which at time t = 0 has non-zero Fourier coefficient only at the six wavenumber $\pm\boldsymbol{\kappa}_{\mathrm{a}}$, $\pm\boldsymbol{\kappa}_{\mathrm{b}}$, and $\pm\boldsymbol{\kappa}_{\mathrm{c}}$. Consider the initial evolution of the velocity field governed by the Euler equations. Use Eq. (6.146) with ν = 0 to show that, at t = 0;

(a)

$$\frac{\mathrm{d}a_j}{\mathrm{d}t} = -i\kappa_l^{\mathrm{a}} P_{jk}\left(\boldsymbol{\kappa}^{\mathrm{a}}\right)\left(b_k^* c_l^* + c_k^* b_l^*\right) \tag{2}$$

(b)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\mathbf{a}\cdot\mathbf{a}^*\right) = -\mathrm{im}\left(\mathbf{a}\cdot\mathbf{b}\times\boldsymbol{\kappa}^{\mathrm{a}}\cdot\mathbf{c} + \mathbf{a}\cdot\mathbf{c}\times\boldsymbol{\kappa}^{\mathrm{a}}\cdot\mathbf{b}\right) \tag{3}$$

where im(z) function returns the imaginary part of a complex value z.

(c)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{a}\cdot\mathbf{a}^* + \mathbf{b}\cdot\mathbf{b}^* + \mathbf{c}\cdot\mathbf{c}^*\right) = 0 \tag{4}$$

(d) there are 24 modes with non-zero rates of change. Sketch their locations in wavenumber space.

Solution

(a)

Eq. (6.146) with ν = 0 and $\boldsymbol{\kappa} = \boldsymbol{\kappa}^{\mathrm{a}}$ is expressed as

$$
\begin{aligned}
\frac{\mathrm{d}a_j}{\mathrm{d}t} &= -i\kappa_l^a P_{jk}\left(\boldsymbol{\kappa}^{\mathrm{a}}\right)\sum_{\boldsymbol{\kappa}'}\hat{u}_k\left(\boldsymbol{\kappa}',t\right)\hat{u}_l\left(\boldsymbol{\kappa}^{\mathrm{a}}-\boldsymbol{\kappa}',t\right) \\
&= -i\kappa_l^a P_{jk}\left(\boldsymbol{\kappa}^{\mathrm{a}}\right)\Big[\hat{u}_k\left(-\boldsymbol{\kappa}^{\mathrm{a}},t\right)\hat{u}_l\left(2\boldsymbol{\kappa}^{\mathrm{a}},t\right)+\hat{u}_k\left(\boldsymbol{\kappa}^{\mathrm{a}},t\right)\hat{u}_l\left(0,t\right) \\
&\quad +\hat{u}_k\left(-\boldsymbol{\kappa}^{\mathrm{b}},t\right)\hat{u}_l\left(\boldsymbol{\kappa}^{\mathrm{a}}+\boldsymbol{\kappa}^{\mathrm{b}},t\right)+\hat{u}_k\left(\boldsymbol{\kappa}^{\mathrm{b}},t\right)\hat{u}_l\left(\boldsymbol{\kappa}^{\mathrm{a}}-\boldsymbol{\kappa}^{\mathrm{b}},t\right) \\
&\quad +\hat{u}_k\left(-\boldsymbol{\kappa}^{\mathrm{c}},t\right)\hat{u}_l\left(\boldsymbol{\kappa}^{\mathrm{a}}+\boldsymbol{\kappa}^{\mathrm{c}},t\right)+\hat{u}_k\left(\boldsymbol{\kappa}^{\mathrm{c}},t\right)\hat{u}_l\left(\boldsymbol{\kappa}^{\mathrm{a}}-\boldsymbol{\kappa}^{\mathrm{c}},t\right)\Big]
\end{aligned} \tag{5}
$$

Using Eq. (6.121) and recall that at t = 0 only the specified six wavenumbers have corresponding Fourier coefficients, we could write

$$
\begin{aligned}
\frac{\mathrm{d}a_j}{\mathrm{d}t} &= -i\kappa_l^a P_{jk}\left(\boldsymbol{\kappa}^{\mathrm{a}}\right)\Big[\hat{u}_k\left(-\boldsymbol{\kappa}^{\mathrm{b}},t\right)\hat{u}_l\left(-\boldsymbol{\kappa}^{\mathrm{c}},t\right)+\hat{u}_k\left(-\boldsymbol{\kappa}^{\mathrm{c}},t\right)\hat{u}_l\left(-\boldsymbol{\kappa}^{\mathrm{b}},t\right)\Big] \\
&= -i\kappa_l^a P_{jk}\left(\boldsymbol{\kappa}^{\mathrm{a}}\right)\Big[b_k^*\left(\boldsymbol{\kappa}^{\mathrm{b}},t\right)c_l^*\left(\boldsymbol{\kappa}^{\mathrm{c}},t\right)+c_k^*\left(\boldsymbol{\kappa}^{\mathrm{c}},t\right)b_l^*\left(\boldsymbol{\kappa}^{\mathrm{b}},t\right)\Big]
\end{aligned} \tag{6}
$$

(b)

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\mathbf{a}\cdot\mathbf{a}^*\right)=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}a_j a_j^*\right)=\frac{1}{2}a_j^*\frac{\mathrm{d}}{\mathrm{d}t}\left(a_j\right)+\frac{1}{2}a_j\frac{\mathrm{d}}{\mathrm{d}t}\left(a_j^*\right) \tag{7}
$$

Introduce the projection tensor

$$
P_{jk}=\delta_{jk}-\frac{\kappa_j\kappa_k}{\kappa^2} \tag{8}
$$

Substitute Eq. (8) into Eq. (6) and we obtain

$$
\begin{aligned}
\frac{\mathrm{d}a_j}{\mathrm{d}t} &= -i\kappa_l^{\mathrm{a}} P_{jk}\left(\boldsymbol{\kappa}^{\mathrm{a}}\right)b_k^*\left(\boldsymbol{\kappa}^{\mathrm{b}}\right)c_l^*\left(\boldsymbol{\kappa}^{\mathrm{c}}\right)-i\kappa_l^a P_{jk}\left(\boldsymbol{\kappa}^{\mathrm{a}}\right)c_k^*\left(\boldsymbol{\kappa}^{\mathrm{c}}\right)b_l^*\left(\boldsymbol{\kappa}^{\mathrm{b}}\right) \\
&= -i\kappa_l^{\mathrm{a}}\left(\delta_{jk}-\frac{\kappa_j^{\mathrm{a}}\kappa_k^{\mathrm{a}}}{\left(\kappa^{\mathrm{a}}\right)^2}\right)b_k^*c_l^*-i\kappa_l^a\left(\delta_{jk}-\frac{\kappa_j^{\mathrm{a}}\kappa_k^{\mathrm{a}}}{\left(\kappa^{\mathrm{a}}\right)^2}\right)c_k^*b_l^* \\
&= -i\kappa_l^{\mathrm{a}}b_j^*c_l^*-i\kappa_l^{\mathrm{a}}\left(-\frac{\kappa_j^{\mathrm{a}}\kappa_k^{\mathrm{a}}}{\left(\kappa^{\mathrm{a}}\right)^2}\right)b_k^*c_l^*-ic_j^*\boldsymbol{\kappa}^{\mathrm{a}}\cdot\mathbf{b}^*-i\kappa_l^a\left(-\frac{\kappa_j^{\mathrm{a}}\kappa_k^{\mathrm{a}}}{\left(\kappa^{\mathrm{a}}\right)^2}\right)c_k^*b_l^* \\
&= -ib_j^*\boldsymbol{\kappa}^{\mathrm{a}}\cdot\mathbf{c}^*+i\frac{\kappa_j^{\mathrm{a}}}{\left(\kappa^{\mathrm{a}}\right)^2}\boldsymbol{\kappa}^{\mathrm{a}}\cdot\mathbf{c}^*\times\boldsymbol{\kappa}^{\mathrm{a}}\cdot\mathbf{b}^*-ic_j^*\boldsymbol{\kappa}^{\mathrm{a}}\cdot\mathbf{b}^*+i\frac{\kappa_j^{\mathrm{a}}}{\left(\kappa^{\mathrm{a}}\right)^2}\boldsymbol{\kappa}^{\mathrm{a}}\cdot\mathbf{b}^*\times\boldsymbol{\kappa}^{\mathrm{a}}\cdot\mathbf{c}^*
\end{aligned} \tag{9}
$$

multiply a_j^* on the both sides of Eq. (9)

$$a_j^* \frac{\mathrm{d}a_j}{\mathrm{d}t} = -ia_j^* b_j^* \boldsymbol{\kappa}^{\mathrm{a}} \cdot \mathbf{c}^* + i\frac{a_j^* \kappa_j^{\mathrm{a}}}{\left(\kappa^{\mathrm{a}}\right)^2} \boldsymbol{\kappa}^{\mathrm{a}} \cdot \mathbf{c}^* \times \boldsymbol{\kappa}^{\mathrm{a}} \cdot \mathbf{b}^*$$

$$-ia_j^* c_j^* \boldsymbol{\kappa}^{\mathrm{a}} \cdot \mathbf{b}^* + i\frac{a_j^* \kappa_j^{\mathrm{a}}}{\left(\kappa^{\mathrm{a}}\right)^2} \boldsymbol{\kappa}^{\mathrm{a}} \cdot \mathbf{b}^* \times \boldsymbol{\kappa}^{\mathrm{a}} \cdot \mathbf{c}^* \tag{10}$$

$$= -i\mathbf{a}^* \cdot \mathbf{b}^* \times \boldsymbol{\kappa}^{\mathrm{a}} \cdot \mathbf{c}^* - i\mathbf{a}^* \cdot \mathbf{c}^* \times \boldsymbol{\kappa}^{\mathrm{a}} \cdot \mathbf{b}^* + i2\frac{\boldsymbol{\kappa}^{\mathrm{a}} \cdot \mathbf{a}^* \times \boldsymbol{\kappa}^{\mathrm{a}} \cdot \mathbf{c}^* \times \boldsymbol{\kappa}^{\mathrm{a}} \cdot \mathbf{b}^*}{\left(\kappa^{\mathrm{a}}\right)^2}$$

With the simple relationship

$$a_j \frac{\mathrm{d}a_j^*}{\mathrm{d}t} = \left(a_j^* \frac{\mathrm{d}a_j}{\mathrm{d}t}\right)^* \tag{11}$$

And note that $(-i\kappa)^* = i\kappa$, we can have

$$a_j \frac{\mathrm{d}a_j^*}{\mathrm{d}t} = i\mathbf{a} \cdot \mathbf{b} \times \boldsymbol{\kappa}^{\mathrm{a}} \cdot \mathbf{c} + i\mathbf{a} \cdot \mathbf{c} \times \boldsymbol{\kappa}^{\mathrm{a}} \cdot \mathbf{b} - i2\frac{\boldsymbol{\kappa}^{\mathrm{a}} \cdot \mathbf{a} \times \boldsymbol{\kappa}^{\mathrm{a}} \cdot \mathbf{c} \times \boldsymbol{\kappa}^{\mathrm{a}} \cdot \mathbf{b}}{\left(\kappa^{\mathrm{a}}\right)^2} \tag{12}$$

Adding Eq. (10) and Eq. (12), the result is

$$a_j^* \frac{\mathrm{d}a_j}{\mathrm{d}t} + a_j \frac{\mathrm{d}a_j^*}{\mathrm{d}t} = -i\mathbf{a}^* \cdot \mathbf{b}^* \times \boldsymbol{\kappa}^{\mathrm{a}} \cdot \mathbf{c}^* - i\mathbf{a}^* \cdot \mathbf{c}^* \times \boldsymbol{\kappa}^{\mathrm{a}} \cdot \mathbf{b}^*$$

$$+ i\mathbf{a} \cdot \mathbf{b} \times \boldsymbol{\kappa}^{\mathrm{a}} \cdot \mathbf{c} + i\mathbf{a} \cdot \mathbf{c} \times \boldsymbol{\kappa}^{\mathrm{a}} \cdot \mathbf{b} \tag{13}$$

$$= -2\mathrm{im}\left(\mathbf{a} \cdot \mathbf{b} \times \boldsymbol{\kappa}^{\mathrm{a}} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{c} \times \boldsymbol{\kappa}^{\mathrm{a}} \cdot \mathbf{b}\right)$$

Thus Eq. (7) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\mathbf{a} \cdot \mathbf{a}^*\right) = -\mathrm{im}\left(\mathbf{a} \cdot \mathbf{b} \times \boldsymbol{\kappa}^{\mathrm{a}} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{c} \times \boldsymbol{\kappa}^{\mathrm{a}} \cdot \mathbf{b}\right) \tag{14}$$

(c)

From Eq. (14), we observe that this expression falls into a cyclic symmetric manner. Then we could write

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{a}\cdot\mathbf{a}^{*}+\mathbf{b}\cdot\mathbf{b}^{*}+\mathbf{c}\cdot\mathbf{c}^{*}\right)=-2\mathrm{im}\big(&\mathbf{a}\cdot\mathbf{b}\times\boldsymbol{\kappa}^{\mathrm{a}}\cdot\mathbf{c}+\mathbf{a}\cdot\mathbf{c}\times\boldsymbol{\kappa}^{\mathrm{a}}\cdot\mathbf{b}\\ &+\mathbf{b}\cdot\mathbf{a}\times\boldsymbol{\kappa}^{\mathrm{b}}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{c}\times\boldsymbol{\kappa}^{\mathrm{b}}\cdot\mathbf{a}\\ &+\mathbf{c}\cdot\mathbf{b}\times\boldsymbol{\kappa}^{\mathrm{c}}\cdot\mathbf{a}+\mathbf{c}\cdot\mathbf{a}\times\boldsymbol{\kappa}^{\mathrm{c}}\cdot\mathbf{b}\big)\\ =-2\mathrm{im}\big(&\mathbf{a}\cdot\mathbf{b}\times\boldsymbol{\kappa}^{\mathrm{a}}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{a}\times\boldsymbol{\kappa}^{\mathrm{b}}\cdot\mathbf{c}\\ &+\mathbf{a}\cdot\mathbf{c}\times\boldsymbol{\kappa}^{\mathrm{a}}\cdot\mathbf{b}+\mathbf{c}\cdot\mathbf{a}\times\boldsymbol{\kappa}^{\mathrm{c}}\cdot\mathbf{b}\\ &+\mathbf{b}\cdot\mathbf{c}\times\boldsymbol{\kappa}^{\mathrm{b}}\cdot\mathbf{a}+\mathbf{c}\cdot\mathbf{b}\times\boldsymbol{\kappa}^{\mathrm{c}}\cdot\mathbf{a}\big)\\ =-2\mathrm{im}\big(&-\mathbf{a}\cdot\mathbf{b}\times\boldsymbol{\kappa}^{\mathrm{c}}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{c}\times\boldsymbol{\kappa}^{\mathrm{b}}\cdot\mathbf{b}-\mathbf{b}\cdot\mathbf{c}\times\boldsymbol{\kappa}^{\mathrm{a}}\cdot\mathbf{a}\big)\end{aligned}\tag{15}$$

Invoke continuity equation, Eq. (6.128), we have

$$\boldsymbol{\kappa}^{\mathrm{a}}\cdot\mathbf{a}=\boldsymbol{\kappa}^{\mathrm{b}}\cdot\mathbf{b}=\boldsymbol{\kappa}^{\mathrm{c}}\cdot\mathbf{c}=0\tag{16}$$

Then Eq. (15) turns into

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{a}\cdot\mathbf{a}^{*}+\mathbf{b}\cdot\mathbf{b}^{*}+\mathbf{c}\cdot\mathbf{c}^{*}\right)=0\tag{17}$$

(d)

Unfortunately, I do not have a proper solution to part (d). I will check upon this exercise later.