Solution to Ex. 6.15

of *Turbulent Flows* by Stephen B. Pope, 2000

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Show that the Fourier coefficient $\hat{\boldsymbol{\omega}}(\boldsymbol{\kappa})$ of the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$

$$\hat{\boldsymbol{\omega}}(\boldsymbol{\kappa}) = F_{\boldsymbol{\kappa}}\{\boldsymbol{\omega}(\mathbf{x})\} = i\boldsymbol{\kappa} \times \hat{\mathbf{u}}(\boldsymbol{\kappa}) \tag{1}$$

Show that $\boldsymbol{\kappa}$, $\hat{\mathbf{u}}(\boldsymbol{\kappa})$, and $\hat{\boldsymbol{\omega}}(\boldsymbol{\kappa})$ are mutually orthogonal.

Solution

Based on the definition of vorticity of a vector, $\hat{\boldsymbol{\omega}}(\boldsymbol{\kappa})$ could be expressed as

$$\begin{aligned}
\hat{\boldsymbol{\omega}}(\boldsymbol{\kappa}) &= F_{\boldsymbol{\kappa}}\{(\nabla \times \mathbf{u})\} \\
&= F_{\boldsymbol{\kappa}}\left\{\left[\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right]^{\mathrm{T}}\right\} \\
&= \left[i\kappa_2\hat{u}_3 - i\kappa_3\hat{u}_2, i\kappa_3\hat{u}_1 - i\kappa_1\hat{u}_3, i\kappa_1\hat{u}_2 - i\kappa_2\hat{u}_1\right]^{\mathrm{T}} \\
&= i\boldsymbol{\kappa} \times \hat{\mathbf{u}}(\boldsymbol{\kappa})
\end{aligned} \tag{2}$$

For incompressible flow, the divergence of velocity field is zero, then

$$F_{\boldsymbol{\kappa}}\{\nabla \cdot \mathbf{u}\} = i\kappa_j\hat{u}_j(\boldsymbol{\kappa}) = i\boldsymbol{\kappa} \cdot \hat{\mathbf{u}}(\boldsymbol{\kappa}) = 0 \tag{3}$$

This indicates that

$$\boldsymbol{\kappa} \cdot \hat{\mathbf{u}}(\boldsymbol{\kappa}) = 0 \tag{4}$$

Thus $\boldsymbol{\kappa}$ and $\hat{\mathbf{u}}(\boldsymbol{\kappa})$ are orthogonal. Recall that Eq. (2) holds true, which means $\hat{\boldsymbol{\omega}}(\boldsymbol{\kappa})$ is orthogonal to both $\boldsymbol{\kappa}$ and $\hat{\mathbf{u}}(\boldsymbol{\kappa})$. And the above indicates that the three vectors are mutually orthogonal.