Solution to Ex. 6.12

of *Turbulent Flows* by Stephen B. Pope, 2000

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Show that, for integer *n*,

$$\int_0^L e^{2\pi inx/L}\mathrm{d}x = \begin{cases} 0, & \text{for } n \neq 0 \\ L, & \text{for } n = 0 \end{cases} \tag{1}$$

and hence establish the orthonormality property Eq. (6.112).

Solution

The integral could be written as sines and cosines

$$\begin{aligned} \int_0^L e^{2\pi inx/L}\mathrm{d}x &= \int_0^L \cos\left(\frac{2\pi nx}{L}\right) + i\sin\left(\frac{2\pi nx}{L}\right)\mathrm{d}x \\ &= \int_0^L \cos\left(\frac{2\pi nx}{L}\right)\mathrm{d}x + i\int_0^L \sin\left(\frac{2\pi nx}{L}\right)\mathrm{d}x \end{aligned} \tag{2}$$

It is obvious that Eq. (1) holds.

Eq. (6.112) is

$$\begin{aligned} \left\langle e^{i\boldsymbol{\kappa}\cdot\mathbf{x}} e^{-i\boldsymbol{\kappa}'\cdot\mathbf{x}} \right\rangle_L &= \frac{1}{L^3}\int_0^L\int_0^L\int_0^L e^{i\boldsymbol{\kappa}\cdot\mathbf{x}} e^{-i\boldsymbol{\kappa}'\cdot\mathbf{x}}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \\ &= \frac{1}{L^3}\int_0^L\int_0^L\int_0^L e^{i(\boldsymbol{\kappa}-\boldsymbol{\kappa}')\cdot\mathbf{x}}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \\ &\overset{\boldsymbol{\kappa}''=\boldsymbol{\kappa}-\boldsymbol{\kappa}'}{=} \frac{1}{L^3}\int_0^L\int_0^L\int_0^L e^{i\boldsymbol{\kappa}''\cdot\mathbf{x}}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \end{aligned} \tag{3}$$

The wavenumber vector **κ**” could be written as

$$\boldsymbol{\kappa}'' = \kappa_0\mathbf{n} = \frac{2\pi}{L}\mathbf{n} \tag{4}$$

where **n** is an integer vector. Substitute Eq. (4) into Eq. (3) we have

$$
\begin{aligned}
\left\langle e^{i\boldsymbol{\kappa}\cdot\mathbf{x}} e^{-i\boldsymbol{\kappa}'\cdot\mathbf{x}} \right\rangle_L &= \frac{1}{L^3}\int_0^L\int_0^L\int_0^L e^{i\frac{2\pi}{L}\mathbf{n}\cdot\mathbf{x}} \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \\
&= \frac{1}{L^3}\int_0^L\int_0^L\int_0^L e^{i\frac{2\pi}{L}n_1x_1} e^{i\frac{2\pi}{L}n_2x_2} e^{i\frac{2\pi}{L}n_3x_3} \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \\
&= \frac{1}{L}\int_0^L e^{i\frac{2\pi}{L}n_1x_1}\mathrm{d}x_1 \frac{1}{L}\int_0^L e^{i\frac{2\pi}{L}n_2x_2}\mathrm{d}x_2 \frac{1}{L}\int_0^L e^{i\frac{2\pi}{L}n_3x_3}\mathrm{d}x_3
\end{aligned}
\qquad (5)
$$

When $\boldsymbol{\kappa} = \boldsymbol{\kappa}'$, $\boldsymbol{\kappa}'' = 0$, then Eq. (5) equals 1. When $\boldsymbol{\kappa} \neq \boldsymbol{\kappa}'$, then Eq. (5) is 0.