Ex. 13.9

of *Turbulent Flows* by Stephen B. Pope, 2000

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Show that the Fourier transform of $\tilde{u}\left(x+\frac{1}{2}h\right)$ is

$$F\left\{\tilde{u}\left(x+\frac{1}{2}h\right)\right\}=e^{i\kappa h/2}F\left\{\tilde{u}\left(x\right)\right\} \tag{1}$$

Hence show that the Fourier transform of $D_h\tilde{u}\left(x\right)$ (Eq. (13.56)) is

$$F\left\{D_h\tilde{u}\left(x\right)\right\}=\frac{i\sin\left(\frac{1}{2}\kappa h\right)}{\frac{1}{2}h}F\left\{\tilde{u}\left(x\right)\right\}=\frac{\sin\left(\frac{1}{2}\kappa h\right)}{\frac{1}{2}\kappa h}F\left\{\frac{\mathrm{d}\tilde{u}\left(x\right)}{\mathrm{d}x}\right\} \tag{2}$$

and verify Eq. (13.57)

Solution

The Fourier transform of $\tilde{u}\left(x+\frac{1}{2}h\right)$ is

$$\begin{aligned}F\left\{\tilde{u}\left(x+\frac{1}{2}h\right)\right\}&=\int_{-\infty}^{+\infty}\tilde{u}\left(x+\frac{1}{2}h\right)e^{-i\kappa x}\mathrm{d}x\\&\overset{t=x+\frac{1}{2}h}{=}\int_{-\infty}^{+\infty}\tilde{u}\left(t\right)e^{-i\kappa\left(t-\frac{1}{2}h\right)}\mathrm{d}t\\&=e^{\frac{i\kappa h}{2}}\int_{-\infty}^{+\infty}\tilde{u}\left(t\right)e^{-i\kappa t}\mathrm{d}t\\&=e^{\frac{i\kappa h}{2}}F\left\{\tilde{u}\left(x\right)\right\}\end{aligned} \tag{3}$$

Based on Eq. (3), the Fourier transform of $D_h\tilde{u}\left(x\right)$ is

$$
\begin{aligned}
F\left\{D_h\tilde{u}(x)\right\} &= F\left\{\frac{\tilde{u}\left(x+\frac{1}{2}h\right)-\tilde{u}\left(x-\frac{1}{2}h\right)}{h}\right\} \\
&= \frac{1}{h}\left[e^{\frac{i\kappa h}{2}}F\left\{\tilde{u}(x)\right\}-e^{-\frac{i\kappa h}{2}}F\left\{\tilde{u}(x)\right\}\right] \\
&= \frac{1}{h}\left(\cos\left(\frac{\kappa h}{2}\right)+i\sin\left(\frac{\kappa h}{2}\right)-\cos\left(\frac{\kappa h}{2}\right)+i\sin\left(\frac{\kappa h}{2}\right)\right)F\left\{\tilde{u}(x)\right\} \\
&= \frac{i\sin\left(\frac{\kappa h}{2}\right)}{\frac{h}{2}}F\left\{\tilde{u}(x)\right\}
\end{aligned}
\tag{4}
$$

Further

$$
\begin{aligned}
F\left\{D_h\tilde{u}(x)\right\} &= \frac{i\sin\left(\frac{\kappa h}{2}\right)}{\frac{h}{2}}F\left\{\tilde{u}(x)\right\} \\
&= \frac{i\sin\left(\frac{\kappa h}{2}\right)}{\frac{h}{2}}(i\kappa)^{-1}F\left\{\frac{\mathrm{d}\tilde{u}(x)}{\mathrm{d}x}\right\} \\
&= \frac{\sin\left(\frac{\kappa h}{2}\right)}{\frac{\kappa h}{2}}F\left\{\frac{\mathrm{d}\tilde{u}(x)}{\mathrm{d}x}\right\}
\end{aligned}
\tag{5}
$$