Ex. 13.8

of *Turbulent Flows* by Stephen B. Pope, 2000

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Suppose that $u(x)$ has the Kolmogorov spectrum $E_{11}(\kappa)$ given by Eq. (6.240) on page 231. Show that, for the Gaussian filter, the spectrum of $\mathrm{d}\bar{u}/\mathrm{d}x$ is

$$\kappa^2 \bar{E}_{11}(\kappa) = C_1 \varepsilon^{2/3} \kappa^{1/3} \exp\left(-\frac{\pi^2 \kappa^2}{12\kappa_\mathrm{c}^2}\right) \tag{1}$$

If $u(x)$ is represented by it Fourier coefficients up to wavenumber κ_r, show that the fraction of $[\mathrm{d}\bar{u}/\mathrm{d}x]^2$ resolved is

$$\frac{\int_0^{\kappa_\mathrm{r}} \kappa^2 \bar{E}_{11}(\kappa)\mathrm{d}\kappa}{\int_0^{\infty} \kappa^2 \bar{E}_{11}(\kappa)\mathrm{d}\kappa} = \frac{\int_0^{(\pi^2/12)(\kappa_\mathrm{r}/\kappa_\mathrm{c})^2} t^{-1/3} e^{-t}\mathrm{d}t}{\int_0^{\infty} t^{-1/3} e^{-t}\mathrm{d}t} = P\left(\frac{2}{3}, \frac{\pi^2}{12}\left(\frac{\kappa_\mathrm{r}}{\kappa_\mathrm{c}}\right)^2\right) \tag{2}$$

where P is the incomplete gamma function. Hence show that, for $\kappa_\mathrm{c}/\kappa_\mathrm{r} = h/\Delta = 1/2$ and 1, this fraction is 0.98 and 0.72, respectively.

Solution

Kolmogorov spectrum is

$$E_{11}(\kappa) = C_1 \varepsilon^{2/3} \kappa^{-5/3} \tag{3}$$

From Eq. (13.37) and for the Gaussian filter, the spectrum of $[\mathrm{d}\bar{u}/\mathrm{d}x]^2$ is

$$\begin{aligned}
\kappa^2 \bar{E}_{11}(\kappa) &= \kappa^2 \left|G(\kappa)\right|^2 E_{11}(\kappa) \\
&= \kappa^2 \left|\exp\left(-\frac{\pi^2\kappa^2}{24\kappa_\mathrm{c}^2}\right)\right|^2 C_1 \varepsilon^{2/3} \kappa^{-5/3} \\
&= C_1 \varepsilon^{2/3} \kappa^{1/3} \exp\left(-\frac{\pi^2 \kappa^2}{12\kappa_\mathrm{c}^2}\right)
\end{aligned} \tag{4}$$

The resolved fraction is

$$
\begin{aligned}
&\frac{\int_0^{\kappa_\mathrm{r}} \kappa^2 \bar{E}_{11}(\kappa)\,\mathrm{d}\kappa}{\int_0^{\infty} \kappa^2 \bar{E}_{11}(\kappa)\,\mathrm{d}\kappa} \\
&= \frac{\int_0^{\kappa_\mathrm{r}} C_1 \varepsilon^{2/3} \kappa^{1/3} \exp\left(-\frac{\pi^2\kappa^2}{12\kappa_\mathrm{c}^2}\right)\mathrm{d}\kappa}{\int_0^{\infty} C_1 \varepsilon^{2/3} \kappa^{1/3} \exp\left(-\frac{\pi^2\kappa^2}{12\kappa_\mathrm{c}^2}\right)\mathrm{d}\kappa} \\
&= \frac{\int_0^{\kappa_\mathrm{r}} \kappa^{1/3} \exp\left(-\frac{\pi^2\kappa^2}{12\kappa_\mathrm{c}^2}\right)\mathrm{d}\kappa}{\int_0^{\infty} \kappa^{1/3} \exp\left(-\frac{\pi^2\kappa^2}{12\kappa_\mathrm{c}^2}\right)\mathrm{d}\kappa}
\end{aligned}
\tag{5}
$$

For the numerator term of Eq. (5) we take a variable substitute

$$
t = \frac{\pi^2\kappa^2}{12\kappa_\mathrm{c}^2} \Rightarrow \kappa = \left(\frac{12\kappa_\mathrm{c}^2}{\pi^2} t\right)^{\frac{1}{2}} \Rightarrow \mathrm{d}\kappa = \frac{6\kappa_\mathrm{c}^2}{\pi^2}\left(\frac{12\kappa_\mathrm{c}^2}{\pi^2} t\right)^{-\frac{1}{2}} \mathrm{d}t
\tag{6}
$$

Apply Eq. (6) to the numerator term of Eq. (5) we obtain

$$
\begin{aligned}
&\int_0^{\kappa_\mathrm{r}} \kappa^{1/3} \exp\left(-\frac{\pi^2\kappa^2}{12\kappa_\mathrm{c}^2}\right)\mathrm{d}\kappa \\
&= \int_0^{\frac{\pi^2\kappa_\mathrm{r}^2}{12\kappa_\mathrm{c}^2}} \left(\left(\frac{12\kappa_\mathrm{c}^2}{\pi^2} t\right)^{\frac{1}{2}}\right)^{1/3} e^{-t}\, \frac{6\kappa_\mathrm{c}^2}{\pi^2}\left(\frac{12\kappa_\mathrm{c}^2}{\pi^2} t\right)^{-\frac{1}{2}} \mathrm{d}t \\
&= \frac{6\kappa_\mathrm{c}^2}{\pi^2}\left(\frac{12\kappa_\mathrm{c}^2}{\pi^2}\right)^{-\frac{1}{3}} \int_0^{\frac{\pi^2\kappa_\mathrm{r}^2}{12\kappa_\mathrm{c}^2}} t^{-\frac{1}{3}} e^{-t}\mathrm{d}t
\end{aligned}
\tag{7}
$$

Substituting Eq. (7) into Eq. (5) we have

$$
\frac{\int_0^{\kappa_\mathrm{r}} \kappa^{1/3} \exp\left(-\frac{\pi^2\kappa^2}{12\kappa_\mathrm{c}^2}\right)\mathrm{d}\kappa}{\int_0^{\infty} \kappa^{1/3} \exp\left(-\frac{\pi^2\kappa^2}{12\kappa_\mathrm{c}^2}\right)\mathrm{d}\kappa}
= \frac{\frac{6\kappa_\mathrm{c}^2}{\pi^2}\left(\frac{12\kappa_\mathrm{c}^2}{\pi^2}\right)^{-\frac{1}{3}} \int_0^{\frac{\pi^2\kappa_\mathrm{r}^2}{12\kappa_\mathrm{c}^2}} t^{-\frac{1}{3}} e^{-t}\mathrm{d}t}{\frac{6\kappa_\mathrm{c}^2}{\pi^2}\left(\frac{12\kappa_\mathrm{c}^2}{\pi^2}\right)^{-\frac{1}{3}} \int_0^{\infty} t^{-\frac{1}{3}} e^{-t}\mathrm{d}t}
= \frac{\int_0^{\frac{\pi^2\kappa_\mathrm{r}^2}{12\kappa_\mathrm{c}^2}} t^{-\frac{1}{3}} e^{-t}\mathrm{d}t}{\int_0^{\infty} t^{-\frac{1}{3}} e^{-t}\mathrm{d}t}
\tag{8}
$$

Introduce gamma function and upper incomplete gamma function

$$\Gamma(s)=\int_0^{\infty} t^{s-t}e^{-t}\mathrm{d}t$$
$$\Gamma(s,x)=\int_0^{x} t^{s-t}e^{-t}\mathrm{d}t \tag{9}$$

Eq. (8) could be written as

$$\frac{\int_0^{\frac{\pi^2\kappa_r^2}{12\kappa_c^2}} t^{-\frac{1}{3}}e^{-t}\mathrm{d}t}{\int_0^{\infty} t^{-\frac{1}{3}}e^{-t}\mathrm{d}t}=\frac{\Gamma\left(\frac{2}{3},\frac{\pi^2\kappa_r^2}{12\kappa_c^2}\right)}{\Gamma\left(\frac{2}{3}\right)}=P\left(\frac{2}{3},\frac{\pi^2\kappa_r^2}{12\kappa_c^2}\right) \tag{10}$$

It fact, *P* is regularized Gamma functions[1]. However, the incomplete gamma function defined in MATLAB is the same with *P*. That is

$$P(s,x)=\frac{\Gamma(s,x)}{\Gamma(s)} \tag{11}$$

For $\kappa_r/\kappa_c=1/2$ and 1, invoking the gammainc() function of MATLAB we get

```
>> gammainc(pi^2/3, 2/3)
ans =
  0.9829
>> gammainc(pi^2/12, 2/3)
ans =
  0.7207
```

References

[1] Anon. Incomplete gamma function[OL]. [02/08/2017] https://en.wikipedia.org/w/index.php?title=Incomplete_gamma_function&oldid=764380939.