Ex. 13.7

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Show that the autocovariance of the filtered fluctuation is

$$\bar{R}(r) \equiv \langle \bar{u}(x+r)\bar{u}(x) \rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(y)G(z)R(r+z-y)\mathrm{d}y\mathrm{d}z \tag{1}$$

Show that the spectrum of $\bar{u}(x)$ can be written

$$\begin{aligned}\bar{E}_{11}(\kappa) &\equiv \frac{1}{\pi}\int_{-\infty}^{+\infty} \bar{R}(r)e^{-i\kappa r}\mathrm{d}r \\ &= \frac{1}{\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(y)e^{-i\kappa y}G(z)e^{i\kappa z}R(r+z-y)e^{-i\kappa(r+z-y)}\mathrm{d}y\mathrm{d}z\mathrm{d}r\end{aligned} \tag{2}$$

For fixed *y* and *z* show

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} R(r+z-y)e^{-i\kappa(r+z-y)}\mathrm{d}r = E_{11}(\kappa) \tag{3}$$

and hence obtain the result

$$\bar{E}_{11}(\kappa) = G(\kappa)G^{*}(\kappa)E_{11}(\kappa) = \left|G(\kappa)\right|^{2} E_{11}(\kappa) \tag{4}$$

Solution

The autocovariance of the filtered fluctuation is

$$\begin{aligned}&\langle \bar{u}(x+r)\bar{u}(x) \rangle \\ &= \left\langle \int_{-\infty}^{+\infty} G(y)u(x+r-y)\mathrm{d}y \int_{-\infty}^{+\infty} G(z)u(x-z)\mathrm{d}z \right\rangle \\ &= \left\langle \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(y)G(z)u(x+r-y)u(x-z)\mathrm{d}y\mathrm{d}z \right\rangle \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(y)G(z)\langle u(x+r-y)u(x-z)\rangle \mathrm{d}y\mathrm{d}z \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(y)G(z)R(r+z-y)\mathrm{d}y\mathrm{d}z\end{aligned} \tag{5}$$

The spectrum of $\bar{u}(x)$ is

$$\begin{aligned}
&\frac{1}{\pi}\int_{-\infty}^{+\infty}\bar{R}(r)e^{-i\kappa r}\mathrm{d}r \\
&=\frac{1}{\pi}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}G(y)G(z)R(r+z-y)\mathrm{d}y\mathrm{d}z\right)e^{-i\kappa r}\mathrm{d}r \\
&=\frac{1}{\pi}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}G(y)G(z)R(r+z-y)\mathrm{d}y\mathrm{d}z\right)e^{-i\kappa(r+z-y)}\mathrm{e}^{i\kappa z}\mathrm{e}^{-i\kappa y}\mathrm{d}r \\
&=\frac{1}{\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}G(y)\mathrm{e}^{-i\kappa y}G(z)\mathrm{e}^{i\kappa z}R(r+z-y)e^{-i\kappa(r+z-y)}\mathrm{d}y\mathrm{d}z\mathrm{d}r
\end{aligned} \tag{6}$$

For fixed y and z, it is straight forward that

$$\begin{aligned}
E_{11}(\kappa)&=\frac{1}{\pi}\int_{-\infty}^{+\infty}R(r)e^{-i\kappa r}\mathrm{d}r \\
&=\frac{1}{\pi}\int_{-\infty}^{+\infty}R(r+z-y)e^{-i\kappa(r+z-y)}\mathrm{d}(r+z-y) \\
&=\frac{1}{\pi}\int_{-\infty}^{+\infty}R(r+z-y)e^{-i\kappa(r+z-y)}\mathrm{d}r
\end{aligned} \tag{7}$$

Alternating the order of the terms in Eq. (6) and we can write

$$\begin{aligned}
\bar{E}_{11}(\kappa)&=\frac{1}{\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}G(y)\mathrm{e}^{-i\kappa y}G(z)\mathrm{e}^{i\kappa z}R(r+z-y)e^{-i\kappa(r+z-y)}\mathrm{d}y\mathrm{d}z\mathrm{d}r \\
&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}G(y)\mathrm{e}^{-i\kappa y}G(z)\mathrm{e}^{i\kappa z}\underbrace{\left(\frac{1}{\pi}\int_{-\infty}^{+\infty}R(r+z-y)e^{-i\kappa(r+z-y)}\mathrm{d}r\right)}_{*}\mathrm{d}y\mathrm{d}z
\end{aligned} \tag{8}$$

For the terms marked by * in Eq. (8), y and z could be assumed to be constant with respect to r. Substituting Eq. (7) into Eq. (8), it follows

$$\begin{aligned}
\bar{E}_{11}(\kappa)&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}G(y)\mathrm{e}^{-i\kappa y}G(z)\mathrm{e}^{i\kappa z}E_{11}(\kappa)\mathrm{d}y\mathrm{d}z \\
&=\int_{-\infty}^{+\infty}G(y)\mathrm{e}^{-i\kappa y}\mathrm{d}y\int_{-\infty}^{+\infty}G(z)\mathrm{e}^{i\kappa z}\mathrm{d}zE_{11}(\kappa) \\
&=\int_{-\infty}^{+\infty}G(y)\mathrm{e}^{-i\kappa y}\mathrm{d}y\int_{-\infty}^{+\infty}G(z)\mathrm{e}^{-i(-\kappa)z}\mathrm{d}zE_{11}(\kappa) \\
&=G(\kappa)G^{*}(\kappa)E_{11}(\kappa) \\
&=\left|G(\kappa)\right|^{2}E_{11}(\kappa)
\end{aligned} \tag{9}$$