Ex. 13.5

Yaoyu Hu
March 12, 2017

Show that the second moment of the filter function is

$$\int_{-\infty}^{+\infty} r^2 G(r)\,\mathrm{d}r = -\left(\frac{\mathrm{d}^2 G(\kappa)}{\mathrm{d}\kappa^2}\right)_{\kappa=0} \tag{1}$$

Hence verify that the second moments of the box and Gaussian filters are $\Delta^2/12$. Comment on the implications for the sharp spectral and Cauchy filters.

Solution

The Fourier transform of G is

$$F\{G(r)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} G(r) e^{-i\kappa r}\mathrm{d}r = \frac{1}{2\pi} G(\kappa) \tag{2}$$

The inverse Fourier transform has the following property

$$\begin{aligned} F^{-1}\left\{\frac{\mathrm{d}^2}{\mathrm{d}\kappa^2}\left(\frac{1}{2\pi}G(\kappa)\right)\right\} &= F^{-1}\left\{\frac{\mathrm{d}^2}{\mathrm{d}\kappa^2}\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} G(r) e^{-i\kappa r}\mathrm{d}r\right)\right\} \\ &= (-ir)^2 G(r) \\ &= -r^2 G(r) \end{aligned} \tag{3}$$

Take the Fourier transform on both sides of Eq. (3) we have

$$F\left\{F^{-1}\left\{\frac{\mathrm{d}^2}{\mathrm{d}\kappa^2}\left(\frac{1}{2\pi}G(\kappa)\right)\right\}\right\} = F\left\{-r^2 G(r)\right\} \tag{4}$$

this means

$$
\begin{gathered}
\frac{\mathrm{d}^2}{\mathrm{d}\kappa^2}\left(\frac{1}{2\pi}G(\kappa)\right)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}-r^2G(r)e^{-i\kappa r}\mathrm{d}r \\
\Rightarrow \\
-\frac{1}{2\pi}\frac{\mathrm{d}^2}{\mathrm{d}\kappa^2}\left(G(\kappa)\right)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}r^2G(r)e^{-i\kappa r}\mathrm{d}r \\
\Rightarrow \\
\int_{-\infty}^{+\infty}r^2G(r)e^{-i\kappa r}\mathrm{d}r=-\frac{\mathrm{d}^2}{\mathrm{d}\kappa^2}\left(G(\kappa)\right)
\end{gathered}
\tag{5}
$$

Let κ be 0 then we have

$$
\int_{-\infty}^{+\infty}r^2G(r)\mathrm{d}r=-\left[\frac{\mathrm{d}^2}{\mathrm{d}\kappa^2}\left(G(\kappa)\right)\right]\Bigg|_{\kappa=0}
\tag{6}
$$

Based on Table 13.2, the second moment of the box filter is

$$
-\left[\frac{\mathrm{d}^2}{\mathrm{d}\kappa^2}\left(\frac{\sin\left(\frac{\Delta}{2}\kappa\right)}{\frac{\Delta}{2}\kappa}\right)\right]\Bigg|_{\kappa=0}=\left[\frac{\Delta^2}{12}\cos\left(\frac{\Delta}{2}\kappa\right)\right]\Bigg|_{\kappa=0}=\frac{\Delta^2}{12}
\tag{7}
$$

And the second moment of the Gaussian filter is

$$
-\left[\frac{\mathrm{d}^2}{\mathrm{d}\kappa^2}\left(e^{-\frac{\Delta^2\kappa^2}{24}}\right)\right]\Bigg|_{\kappa=0}=\left[-e^{-\frac{\Delta^2\kappa^2}{24}}\left(-\frac{\Delta^2\kappa}{12}\right)^2+e^{-\frac{\Delta^2\kappa^2}{24}}\frac{\Delta^2}{12}\right]\Bigg|_{\kappa=0}=\frac{\Delta^2}{12}
\tag{8}
$$

The second moment of Cauchy filter may be negative value and the second moment of sharp spectral filter may be zero.