# Ex. 13.4

Yaoyu Hu
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Let <>$_\Delta$ denote the operation of applying the Gaussian filter of width $\Delta$, i.e.,

$$\left\langle U(x) \right\rangle_\Delta \equiv \int_{-\infty}^{+\infty} \left( \frac{6}{\pi \Delta^2} \right)^{\frac{1}{2}} \exp\left( \frac{-6r^2}{\Delta^2} \right) U(x-r) \mathrm{d}r \tag{1}$$

and let $G(\kappa;\Delta)$ denote the corresponding transfer function, i.e.,

$$G(\kappa;\Delta) \equiv \exp\left( -\frac{\kappa^2 \Delta^2}{24} \right) \tag{2}$$

Obtain the result

$$G(\kappa;\Delta_\mathrm{a}) G(\kappa;\Delta_\mathrm{b}) = G(\kappa;\Delta_\mathrm{c}) \tag{3}$$

where

$$\Delta_\mathrm{c} = \left( \Delta_\mathrm{a}^2 + \Delta_\mathrm{b}^2 \right)^{\frac{1}{2}} \tag{4}$$

Hence show that

$$\left\langle \left\langle U(x) \right\rangle_\Delta \right\rangle_\Delta = \left\langle U(x) \right\rangle_{\sqrt{2}\Delta} \tag{5}$$

Solution

It is easy to obtain that

$$\begin{aligned} G(\kappa;\Delta_\mathrm{a}) G(\kappa;\Delta_\mathrm{b}) &= \exp\left( -\frac{\kappa^2 \Delta_\mathrm{a}^2}{24} \right) \exp\left( -\frac{\kappa^2 \Delta_\mathrm{b}^2}{24} \right) \\ &= \exp\left( -\frac{\kappa^2 \left( \Delta_\mathrm{a}^2 + \Delta_\mathrm{b}^2 \right)}{24} \right) \end{aligned} \tag{6}$$

thus

$$\Delta_{\mathrm{c}}^2 = \Delta_{\mathrm{a}}^2 + \Delta_{\mathrm{b}}^2 \tag{7}$$

Take the Fourier transform of the left hand side of Eq. (5)

$$\begin{aligned} F\left\{\left\langle\left\langle U\left(x\right)\right\rangle_{\Delta}\right\rangle_{\Delta}\right\} &= G\left(\kappa;\Delta\right)^2 \hat{U}\left(\kappa\right) \\ &= G\left(\kappa;\sqrt{2}\Delta\right)\hat{U}\left(\kappa\right) \end{aligned} \tag{8}$$

The if we apply inverse Fourier transform on both sides of Eq. (8), we have

$$\left\langle\left\langle U\left(x\right)\right\rangle_{\Delta}\right\rangle_{\Delta} = \left\langle U\left(x\right)\right\rangle_{\sqrt{2}\Delta} \tag{9}$$