Ex. 13.3

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For the sharp spectral filter, show that

$$G(\kappa)^2 = G(\kappa) \tag{1}$$

Hence obtain the results (for the sharp spectral filter)

$$\bar{\bar{U}}(x) = \bar{U}(x) \tag{2}$$

$$\bar{u}'(x) = 0 \tag{3}$$

More generally, show that the filtered residual is zero (Eq. (3)) if, and only if, the filtering operation is a projection, i.e., it yields Eq. (2).

Solution

For sharp spectral filter, its transfer function is an *H* function. As far as I know,

$$H(\kappa_c - |\kappa|) = \begin{cases} 1 & |\kappa| < \kappa_c \\ 0 & |\kappa| > \kappa_c \end{cases} \tag{4}$$

and for $|\kappa| = \kappa_c$, *H* = 1/2. So for Eq. (1) to be valid, $|\kappa| = \kappa_c$ should not be included as the definition range of *H*.

Considering Eq. (13.17), we have

$$\hat{\bar{\bar{U}}}(x) = G(\kappa)^2 \hat{U}(\kappa) \tag{5}$$

from Eq. (1) we can write

$$\hat{\bar{\bar{U}}}(x) = G(\kappa)\hat{U}(\kappa) = \hat{\bar{U}}(x) \tag{6}$$

Take inverse Fourier transform on both sides of Eq. (6) we have

$$\bar{\bar{U}}(x) = \bar{U}(x) \tag{7}$$

Since it is true that Eq. (13.18) holds, then Eq. (7) means

$$\bar{u}'(x) = \bar{U}(x) - \bar{\bar{U}}(x) = 0 \tag{8}$$