Solution to Ex. 13.21

of *Turbulent Flows* by Stephen B. Pope, 2000

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For a general filter $G(\mathbf{r}, \mathbf{x})$ satisfying the normalization condition Eq. (13.2), the filtered density function (Pope 1990) is defined by

$$\overline{f}(\mathbf{V};\mathbf{x},t) \equiv \int G(\mathbf{r},\mathbf{x})\delta\left[\mathbf{U}(\mathbf{x}-\mathbf{r},t)-\mathbf{V}\right]\mathrm{d}\mathbf{r} \tag{1}$$

Obtain the results

$$\overline{\mathbf{U}} = \int \mathbf{V}\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V} \tag{2}$$

$$\overline{U_i U_j} = \int V_i V_j \overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V} \tag{3}$$

$$\tau_{ij}^{\mathrm{R}} = \int \left(V_i - \overline{U}_i\right)\left(V_j - \overline{U}_j\right)\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V} \tag{4}$$

where integration is over all $\mathbf{V}$; and $\overline{\mathbf{U}}$, $\overline{U_i U_j}$, and τ_{ij}^{R} are evaluated at $\mathbf{x}$, t. Show that $\overline{f}$ satisfies the normalization condition Eq. (12.1) and that, if the filter is everywhere non-negative, then $\overline{f}$ is also non-negative, and hence has the properties of a joint PDF. Argue that, for such positive filters, the residual stress τ_{ij}^{R} is positive semi-definite (Gao and O'Brien 1993). Use similar reasoning to show that L_{ij}^{o} and R_{ij}^{o} (Eqs. (13.100) and (13.102)) are also positive semi-definite for positive filters.

Solution

$$\begin{aligned}\int \overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V} &\equiv \int\int G(\mathbf{r},\mathbf{x})\delta\left[\mathbf{U}(\mathbf{x}-\mathbf{r},t)-\mathbf{V}\right]\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{V} \\ &= \int G(\mathbf{r},\mathbf{x})\int \delta\left[\mathbf{U}(\mathbf{x}-\mathbf{r},t)-\mathbf{V}\right]\mathrm{d}\mathbf{V}\mathrm{d}\mathbf{r} \\ &= \int G(\mathbf{r},\mathbf{x})\mathrm{d}\mathbf{r} \\ &= 1\end{aligned} \tag{5}$$

This means the filtered density function satisfies the normalization condition. If G(r, x) is non-negative, from Eq. (1) it is clear that $\overline{f}$ is also non-negative.

$$
\begin{aligned}
\int \mathbf{V}\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V} &= \int \mathbf{V}\int G(\mathbf{r},\mathbf{x})\delta\left[\mathbf{U}(\mathbf{x}-\mathbf{r},t)-\mathbf{V}\right]\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{V} \\
&= \int G(\mathbf{r},\mathbf{x})\int \mathbf{V}\delta\left[\mathbf{U}(\mathbf{x}-\mathbf{r},t)-\mathbf{V}\right]\mathrm{d}\mathbf{V}\mathrm{d}\mathbf{r} \\
&= \int G(\mathbf{r},\mathbf{x})\mathbf{U}(\mathbf{x}-\mathbf{r},t)\mathrm{d}\mathbf{r} \\
&= \overline{\mathbf{U}}(\mathbf{x},t)
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
\int V_iV_j\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V} &= \int V_iV_j\int G(\mathbf{r},\mathbf{x})\delta\left[\mathbf{U}(\mathbf{x}-\mathbf{r},t)-\mathbf{V}\right]\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{V} \\
&= \int G(\mathbf{r},\mathbf{x})\int V_iV_j\delta\left[\mathbf{U}(\mathbf{x}-\mathbf{r},t)-\mathbf{V}\right]\mathrm{d}\mathbf{V}\mathrm{d}\mathbf{r} \\
&= \int G(\mathbf{r},\mathbf{x})U_i(\mathbf{x}-\mathbf{r},t)U_j(\mathbf{x}-\mathbf{r},t)\mathrm{d}\mathbf{r} \\
&= \overline{U_i(\mathbf{x},t)U_j(\mathbf{x},t)}
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
&\int \left(V_i-\overline{U}_i\right)\left(V_j-\overline{U}_j\right)\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V} \\
&= \int \left(V_iV_j - V_i\overline{U}_j - \overline{U}_iV_j + \overline{U}_i\overline{U}_j\right)\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V} \\
&= \int V_iV_j\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V} - \overline{U}_j\int V_i\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V} - \overline{U}_i\int V_j\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V} + \overline{U}_i\overline{U}_j\int \overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V} \\
&= \overline{U_iU_j} - \overline{U}_j\overline{U}_i - \overline{U}_i\overline{U}_j + \overline{U}_i\overline{U}_j \\
&= \overline{U_iU_j} - \overline{U}_i\overline{U}_j \\
&= \tau_{ij}^{\mathrm{R}}
\end{aligned} \tag{8}
$$

From the above equations we can write

$$
\begin{aligned}
\tau_{ij}^{\mathrm{R}} &= \overline{U_iU_j} - \overline{U}_i\overline{U}_j \\
&= \int V_iV_j\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V} - \int V_i\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V}\int V_j\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V}
\end{aligned} \tag{9}
$$

For arbitrary vector **Y**, and let $A(\mathbf{V}) = Y_iV_i$.

$$
\begin{aligned}
\tau_{ij}^{\mathrm{R}}Y_iY_j &= Y_iY_j\left(\overline{U_iU_j} - \overline{U}_i\overline{U}_j\right) \\
&= Y_iY_j\int V_iV_j\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V} - Y_iY_j\int V_i\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V}\int V_j\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V} \\
&= \int Y_iV_iY_jV_j\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V} - \int Y_iV_i\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V}\int Y_jV_j\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V} \\
&= \int A^2(\mathbf{V})\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V} - \int A(\mathbf{V})\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V}\int A(\mathbf{V})\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V} \\
&= \int \overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V}\int A^2(\mathbf{V})\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V} - \int A(\mathbf{V})\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V}\int A(\mathbf{V})\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V} \\
&\overset{*}{\geq} \left(\int A(\mathbf{V})\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V}\right)^2 - \int A(\mathbf{V})\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V}\int A(\mathbf{V})\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V} \\
&= 0
\end{aligned}
$$

(10)

where the inequality marked by * is the direct result of the Cauchy-Schwarz inequality[1] and the fact that $\overline{f}$ is positive

$$
\begin{aligned}
&\int \overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V}\int A^2(\mathbf{V})\,\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V} \\
&=\int\left[\sqrt{\overline{f}(\mathbf{V};\mathbf{x},t)}\right]^2\mathrm{d}\mathbf{V}\int\left[A(\mathbf{V})\sqrt{\overline{f}(\mathbf{V};\mathbf{x},t)}\right]^2\mathrm{d}\mathbf{V} \\
&\geq\left[\int\sqrt{\overline{f}(\mathbf{V};\mathbf{x},t)}A(\mathbf{V})\sqrt{\overline{f}(\mathbf{V};\mathbf{x},t)}\mathrm{d}\mathbf{V}\right]^2 \\
&=\int A(\mathbf{V})\,\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V}\int A(\mathbf{V})\,\overline{f}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V}
\end{aligned}
\tag{11}
$$

This derivation is inspired by my colleague DONG Bing <dongbing@sjtu.edu.cn>.

As for Leonard stresses L_{ij}^{o} we can define a new filtered density function

$$
\overline{f^{L}}(\mathbf{V};\mathbf{x},t)\equiv\int G(\mathbf{r},\mathbf{x})\delta\left[\overline{\mathbf{U}}(\mathbf{x}-\mathbf{r},t)-\mathbf{V}\right]\mathrm{d}\mathbf{r} \tag{12}
$$

Then

$$
\begin{aligned}
\int\mathbf{V}\,\overline{f^{L}}(\mathbf{V};\mathbf{x},t)\mathrm{d}\mathbf{V} &=\int\mathbf{V}\int G(\mathbf{r},\mathbf{x})\delta\left[\overline{\mathbf{U}}(\mathbf{x}-\mathbf{r},t)-\mathbf{V}\right]\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{V} \\
&=\int G(\mathbf{r},\mathbf{x})\int\mathbf{V}\delta\left[\overline{\mathbf{U}}(\mathbf{x}-\mathbf{r},t)-\mathbf{V}\right]\mathrm{d}\mathbf{V}\mathrm{d}\mathbf{r} \\
&=\int G(\mathbf{r},\mathbf{x})\overline{\mathbf{U}}(\mathbf{x}-\mathbf{r},t)\mathrm{d}\mathbf{r} \\
&=\overline{\overline{\mathbf{U}}}(\mathbf{x},t)
\end{aligned}
\tag{13}
$$

Here we have two pairs of analogies with $\overline{f^{L}}$ to $\overline{f}$ and $\overline{\overline{\mathbf{U}}}$ to $\overline{\mathbf{U}}$. If we take **U** to be a general random process **B**, then Eq. (10) tells us that a tensor defined by $\overline{B_iB_j}-\overline{B}_i\overline{B}_j=\overline{U_iU_j}-\overline{U}_i\overline{U}_j$ is positive semi-definite. Since L_{ij}^{o} is defined to be

$$
L_{ij}^{o}=\overline{\overline{U}_i\overline{U}_j}-\overline{\overline{U}}_i\overline{\overline{U}}_j \tag{14}
$$

And if we take $\overline{\mathbf{U}}$ to be the general random process **B**, then L_{ij}^{o} must be semi-definite due to the same reason expressed by Eq. (10).

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[1] https://en.wikipedia.org/wiki/Cauchy%E2%80%93Schwarz_inequality

Because the SGS Reynolds stresses are defined in the same way of L_{ij}^{o}

$$R_{ij}^{o} = \overline{u_i' u_j'} - \overline{u'}_i \overline{u'}_j \tag{15}$$

And we take $\mathbf{u}'$ as the general random process **B**, then R_{ij}^{o} is also semi-definite.