Solution to Ex. 13.20

of *Turbulent Flows* by Stephen B. Pope, 2000

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Verify the validity of Germano's decomposition, Eq. (13.99). Show that the Leonard stresses, cross stresses and SGS Reynolds stresses defined by Eqs. (13.100)-(13.102) are Galilean invariant. Show that, if the filter is a projection, then Germano's decomposition (Eqs. (13.99)-(13.102)) is identical to Leonard's (Eq. (13.105)-(13.108)).

Solution

Using the definition of Ex.13.19, for Germano's decomposition, the Leonard stresses are

$$\begin{aligned}
&\overline{\overline{W}_i\overline{W}_j} - \overline{\overline{W}}_i\overline{\overline{W}}_j \\
&= \overline{\left(\overline{U}_i + V_i\right)\left(\overline{U}_j + V_j\right)} - \overline{\left(\overline{U}_i + V_i\right)}\,\overline{\left(\overline{U}_j + V_j\right)} \\
&= \overline{\overline{U}_i\overline{U}_j} + V_j\overline{\overline{U}}_i + V_i\overline{\overline{U}}_j + V_iV_j - \overline{\overline{U}}_i\overline{\overline{U}}_j - V_j\overline{\overline{U}}_i - V_i\overline{\overline{U}}_j - V_iV_j \\
&= \overline{\overline{U}_i\overline{U}_j} - \overline{\overline{U}}_i\overline{\overline{U}}_j \\
&= L_{ij}^o
\end{aligned} \tag{1}$$

The cross stresses are

$$\begin{aligned}
&\overline{\overline{W}_i w'_j} + \overline{w'_i\overline{W}_j} - \overline{\overline{W}}_i\,\overline{w'}_j - \overline{w'}_i\,\overline{\overline{W}}_j \\
&= \overline{\left(\overline{U}_i + V_i\right)u'_j} + \overline{u'_i\left(\overline{U}_j + V_j\right)} - \overline{\left(\overline{U}_i + V_i\right)}\,\overline{u'_j} - \overline{u'_i}\,\overline{\left(\overline{U}_j + V_j\right)} \\
&= \overline{\overline{U}_i u'_j} + V_i\overline{u'_j} + \overline{u'_i\overline{U}_j} + \overline{u'_i}V_j - \overline{\overline{U}}_i\,\overline{u'_j} - V_i\overline{u'_j} - \overline{u'_i}\,\overline{\overline{U}}_j - \overline{u'_i}V_j \\
&= \overline{\overline{U}_i u'_j} + \overline{u'_i\overline{U}_j} - \overline{\overline{U}}_i\,\overline{u'_j} - \overline{u'_i}\,\overline{\overline{U}}_j \\
&= C_{ij}^o
\end{aligned} \tag{2}$$

The SGS Reynolds stresses are

$$\overline{w'_iw'_j} - \overline{w'}_i\,\overline{w'}_j = \overline{u'_iu'_j} - \overline{u'}_i\,\overline{u'}_j = R_{ij}^o \tag{3}$$

Eq. (1) to Eq. (3) all indicate that the Germano's decomposition is Galilean-invariant.

Further if we write the residual stresses in terms of Germano's decomposition and assuming that the filter is a projection

$$
\begin{aligned}
\tau_{ij}^{\mathrm{R}} &= L_{ij}^{o} + C_{ij}^{o} + R_{ij}^{o} \\
&= \overline{\overline{U}_i \overline{U}_j} - \overline{\overline{U}}_i \overline{\overline{U}}_j + \overline{\overline{U}_i u'_j} + \overline{u'_i \overline{U}_j} - \overline{\overline{U}}_i \overline{u'_j} - \overline{u'_i}\overline{\overline{U}}_j + \overline{u'_i u'_j} - \overline{u'}_i \overline{u'}_j \\
&= \overline{\overline{U}_i \overline{U}_j} - \overline{U}_i \overline{U}_j + \overline{\overline{U}_i u'_j} + \overline{u'_i \overline{U}_j} + \overline{u'_i u'_j} \\
&= L_{ij} + C_{ij} + R_{ij}
\end{aligned}
\tag{4}
$$

with the fact that (Eq. (13.20) and Eq. (13.21))

$$\overline{\overline{U}}_i = \overline{U}_i \tag{5}$$

$$\overline{u'}_i = 0 \tag{6}$$