# Ex. 13.2

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Let $U(x)$ have the Fourier transform $\hat{U}(\kappa)$ (Eq. (13.10)), so that $\overline{U}(x)$ has the Fourier transform $\hat{\overline{U}}(\kappa) = G(\kappa)\hat{U}(\kappa)$ (Eq. (13.11)). Show that the Fourier transform of the residual $\hat{u}'(\kappa)$ is

$$\hat{u}'(\kappa) \equiv F\left\{u'(x)\right\} = \left[1 - G(\kappa)\right]\hat{U}(\kappa) \tag{1}$$

that the Fourier transform of the filtered residual $\overline{u}'$ is

$$\hat{\overline{u}}'(\kappa) \equiv F\left\{\overline{u}'(x)\right\} = G(\kappa)\left[1 - G(\kappa)\right]\hat{U}(\kappa) \tag{2}$$

and that the Fourier transform of the doubly filtered field $\overline{\overline{U}}(\mathrm{x})$ is

$$\hat{\overline{\overline{U}}}(x) \equiv \mathrm{F}\left\{\overline{\overline{U}}(x)\right\} = G(\kappa)^2 \hat{U}(\kappa) \tag{3}$$

Show that both Eq. (13.4) and the above equations lead to the result

$$\overline{u}' = \overline{U}(x) - \overline{\overline{U}}(x) \tag{4}$$

## Solution

The Fourier transform of the residual is

$$\begin{aligned}
\hat{u}'(\kappa) &= F\left\{u'(x)\right\} \\
&= F\left\{U(x) - \overline{U}(x)\right\} \\
&= F\left\{U(x)\right\} - F\left\{\overline{U}(x)\right\} \\
&= \hat{U}(\kappa) - G(\kappa)\hat{U}(\kappa) \\
&= \left[1 - G(\kappa)\right]\hat{U}(\kappa)
\end{aligned} \tag{5}$$

From Eq. (13.11) and Eq. (5) it is straight forward that

$$\begin{aligned}\hat{\bar{u}}'(\kappa) &= G(\kappa)\hat{u}'(\kappa) \\ &= G(\kappa)\left[1-G(\kappa)\right]\hat{U}(\kappa)\end{aligned} \tag{6}$$

The Fourier transform of the doubly filtered field is

$$\begin{aligned}\hat{\bar{\bar{U}}}(\kappa) &= F\left\{\bar{\bar{U}}(x)\right\} \\ &= G(\kappa)\hat{\bar{U}}(\kappa) \\ &= G(\kappa)G(\kappa)\hat{U}(\kappa) \\ &= G(\kappa)^2\hat{U}(\kappa)\end{aligned} \tag{7}$$

Eq. (13.3) could be written as Eq. (8) in 1D condition

$$u'(x) = U(x) - \bar{U}(x) \tag{8}$$

Following the definition of filtering operation, we have

$$\begin{aligned}\bar{u}'(x) &= \int G(r)\left(U(x-r) - \bar{U}(x-r)\right)\mathrm{d}r \\ &= \int G(r)U(x-r)\mathrm{d}r - \int G(r)\bar{U}(x-r)\mathrm{d}r \\ &= \bar{U}(x) - \bar{\bar{U}}(x)\end{aligned} \tag{9}$$