Solution to Ex. 13.19

of *Turbulent Flows* by Stephen B. Pope, 2000

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In order to consider the Galilean invariance of the various stresses in the original Leonard decomposition, Eq. (13.105), consider the transformed velocity field

$$\mathbf{W}(\mathbf{x},t) \equiv \mathbf{U}(\mathbf{x},t) + \mathbf{V} \tag{1}$$

where **V** is a constant velocity difference. Obtain the result

$$\overline{W_i W_j} - \overline{W}_i \overline{W}_j = \overline{U_i U_j} - \overline{U}_i \overline{U}_j \tag{2}$$

showing that the residual stress τ_{ij}^{R} is Galilean invariant; and also

$$\overline{\overline{W}_i \overline{W}_j} - \overline{W}_i \overline{W}_j = \overline{\overline{U}_i \overline{U}_j} - \overline{U}_i \overline{U}_j - V_i \overline{u'_j} - V_j \overline{u'_i} \tag{3}$$

showing that (in general) the Leonard stresses are not Galilean-invariant. For the other stresses in Eq. (13.105), show that R_{ij} is Galilean invariant whereas (in general) C_{ij} is not.

Solution

It is straight forward that

$$\overline{W}_i = \overline{U_i + V_i} = \overline{U}_i + V_i \tag{4}$$

$$w'_i = W_i - \overline{W}_i = (U_i + V_i) - (\overline{U}_i + V_i) = U_i - \overline{U}_i = u'_i \tag{5}$$

For the residual stress

$$\begin{aligned}
\overline{W_i W_j} - \overline{W}_i \overline{W}_j &= \overline{(U_i + V_i)(U_j + V_j)} - \overline{(U_i + V_i)}\,\overline{(U_j + V_j)} \\
&= \overline{U_i U_j} + \overline{U_i V_j} + \overline{U_j V_i} + \overline{V_i V_j} - (\overline{U}_i + \overline{V}_i)(\overline{U}_j + \overline{V}_j) \\
&= \overline{U_i U_j} + \overline{U_i V_j} + \overline{U_j V_i} + \overline{V_i V_j} - \overline{U}_i \overline{U}_j - \overline{U}_i \overline{V}_j - \overline{U}_j \overline{V}_i - \overline{V}_i \overline{V}_j \\
&= \overline{U_i U_j} + \overline{U}_i V_j + \overline{U}_j V_i + V_i V_j - \overline{U}_i \overline{U}_j - \overline{U}_i V_j - \overline{U}_j V_i - V_i V_j \\
&= \overline{U_i U_j} - \overline{U}_i \overline{U}_j
\end{aligned} \tag{6}$$

This means τ_{ij}^{R} Galilean-invariance.

$$
\begin{aligned}
&\overline{\overline{W}_i \overline{W}_j} - \overline{W}_i \overline{W}_j \\
&= \overline{\left(\overline{U}_i + V_i\right)\left(\overline{U}_j + V_j\right)} - \left(\overline{U}_i + V_i\right)\left(\overline{U}_j + V_j\right) \\
&= \overline{\overline{U}_i \overline{U}_j + \overline{U}_i V_j + \overline{U}_j V_i + V_i V_j} - \overline{U}_i \overline{U}_j - \overline{U}_i V_j - \overline{U}_j V_i - V_i V \\
&= \overline{\overline{U}_i \overline{U}_j} + \overline{\left(U_i - u_i'\right) V_j} + \overline{\left(U_j - u_j'\right) V_i} + V_i V_j - \overline{U}_i \overline{U}_j - \overline{U}_i V_j - \overline{U}_j V_i - V_i V \\
&= \overline{\overline{U}_i \overline{U}_j} - \overline{U}_i \overline{U}_j - V_i \overline{u_j'} - V_j \overline{u_i'}
\end{aligned}
\tag{7}
$$

Eq. (7) is the Leonard stress based on **W**, and it indicates that Galilean-invariance does not hold. For the cross stress

$$
\begin{aligned}
&\overline{\overline{W}_i u_j'} + \overline{u_i' \overline{W}_j} \\
&= \overline{\left(\overline{U}_i + V_i\right) u_j'} + \overline{u_i' \left(\overline{U}_j + V_j\right)} \\
&= \overline{\overline{U}_i u_j'} + \overline{u_i' \overline{U}_j} + V_i \overline{u_j'} + \overline{u_i'} V_j
\end{aligned}
\tag{8}
$$

It is not Galilean-invariant. For the SGS Reynolds stress, using Eq. (5)

$$
\overline{w_i' w_j'} = \overline{u_i' u_j'} \tag{9}
$$

And the SGS Reynolds stress in Galilean-invariant.