Solution to Ex. 13.18

of *Turbulent Flows* by Stephen B. Pope, 2000

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From the decomposition $\mathbf{U} = \overline{\mathbf{U}} + \mathbf{u}'$, show that (as originally proposed by Leonard (1974)) the residual stress tensor can be decomposed as

$$\tau_{ij}^{\mathrm{R}} \equiv L_{ij} + C_{ij} + R_{ij} \tag{1}$$

where the Leonard stresses are

$$L_{ij} \equiv \overline{\overline{U}_i \overline{U}_j} - \overline{U}_i \overline{U}_j \tag{2}$$

the cross stresses are

$$C_{ij} \equiv \overline{\overline{U}_i u_j'} + \overline{u_i' \overline{U}_j} \tag{3}$$

and the SGS Reynolds stresses are

$$R_{ij} \equiv \overline{u_i' u_j'} \tag{4}$$

(Note that, although the same names are used, these stresses are different than those defined by Eqs. (13.100)-(13.102).) show that τ_{ij}^{R} and τ_{ij}^{κ} differ by the Leonard stress, i.e.,

$$\tau_{ij}^{\mathrm{R}} - \tau_{ij}^{\kappa} = L_{ij} \tag{5}$$

Solution

$$\begin{aligned}
\tau_{ij}^{\mathrm{R}} &= \overline{U_i U_j} - \overline{U}_i \overline{U}_j \\
&= \overline{\left(\overline{U}_i + u_i'\right)\left(\overline{U}_j + u_j'\right)} - \overline{U}_i \overline{U}_j \\
&= \overline{\overline{U}_i \overline{U}_j + \overline{U}_i u' + \overline{U}_j u_i' + u_i' u_j'} - \overline{U}_i \overline{U}_j \\
&= \overline{\overline{U}_i \overline{U}_j} - \overline{U}_i \overline{U}_j + \overline{\overline{U}_i u_j'} + \overline{\overline{U}_j u_i'} + \overline{u_i' u_j'} \\
&= L_{ij} + C_{ij} + R_{ij}
\end{aligned} \tag{6}$$