Solution to Ex. 13.14

of *Turbulent Flows* by Stephen B. Pope, 2000

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Show that, for the sharp spectral filter, the constant af defined by Eq. (13.77) is

$$a_{\mathrm{f}} = \frac{3}{2}\pi^{\frac{4}{3}} \approx 6.90 \tag{1}$$

and that for the Gaussian filter it is

$$a_{\mathrm{f}} = 12^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right) \approx 7.10 \tag{2}$$

Solution

For sharp spectral filter

$$\begin{aligned}
a_{\mathrm{f}} &= 2\int_0^{\infty} (\kappa\Delta)^{\frac{1}{3}} G(\kappa)^2 \, \Delta \mathrm{d}\kappa \\
&= 2\int_0^{\infty} (\kappa\Delta)^{\frac{1}{3}} H\left(\kappa_{\mathrm{c}} - |\kappa|\right)^2 \Delta \mathrm{d}\kappa \\
&= 2\int_0^{\kappa_{\mathrm{c}} = \frac{\pi}{\Delta}} (\kappa\Delta)^{\frac{1}{3}} \Delta \mathrm{d}\kappa \\
&= \frac{3}{2}(\kappa\Delta)^{\frac{4}{3}}\Big|_0^{\frac{\pi}{\Delta}} \\
&= \frac{3}{2}\pi^{\frac{4}{3}}
\end{aligned} \tag{3}$$

For the Gaussian filter

$$
\begin{aligned}
a_{\mathrm{f}} &= 2\int_0^{\infty}\left(\kappa\Delta\right)^{\frac{1}{3}} G\left(\kappa\right)^2 \Delta \mathrm{d}\kappa \\
&= 2\int_0^{\infty}\left(\kappa\Delta\right)^{\frac{1}{3}} e^{-\frac{\kappa^2\Delta^2}{12}} \Delta \mathrm{d}\kappa \\
&\overset{t=\kappa\Delta}{=} 2\int_0^{\infty} t^{\frac{1}{3}} e^{-\frac{t^2}{12}} \mathrm{d}t \\
&= 12^{\frac{2}{3}}\int_0^{\infty}\left(\frac{t^2}{12}\right)^{\frac{2}{3}-1} e^{-\frac{t^2}{12}} \mathrm{d}\frac{t^2}{12} \\
&= 12^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)
\end{aligned} \tag{4}
$$