Solution to Ex. 13.11

of *Turbulent Flows* by Stephen B. Pope, 2000

Yaoyu Hu

March 15, 2017

Consider solving the LES equations by a pseudo-spectral method in which wavenumbers with $|\boldsymbol{\kappa}| < \kappa_r$ are resolved. Argue that, for the Gaussian filter and for $\kappa_r = \kappa_c$ (corresponding to poor spatial resolution), a factor of $(54/38)^3 \approx 2.9$ more nodes is required than would be required if the sharp spectral filter were used (with κ_c chosen to resolve 80% of the energy in each case). Show that the corresponding factor is 23 if the better resolution $\kappa_r = 2\kappa_c$ (h=1/2Δ) is used.

Solution

As discussed in Ex.13.10, for the sharp spectral filter, if 80% energy is required to be resolved κ_c should satisfy

$$\kappa_{c,ss} L \approx 38 \tag{1}$$

where $\kappa_{c,ss}$ is the characteristic wavenumber of the sharp spectral filter. Similarly, for the Gaussian filter, if the same level of energy is required to be resolved, $\kappa_{c,G}$ should satisfy

$$\kappa_{c,G} L \approx 54 \tag{2}$$

We already known that the grid spacing h and grid number N, the filter width Δ, the resolved wavenumber κ_r and the characteristic wavenumber of a filter have the following relationship

$$h \propto \frac{1}{N} \tag{3}$$

$$\kappa_r = \frac{\pi}{h} = \kappa_{\frac{N}{2}} \tag{4}$$

$$\kappa_c = \frac{\pi}{\Delta} \tag{5}$$

$$\frac{h}{\Delta} = \frac{\kappa_c}{\kappa_r} \tag{6}$$

If we choose a poor spatial condition, that is

$$\kappa_{\mathrm{r}} = \kappa_{\mathrm{c}} \tag{7}$$

Then from Eq. (6) we have

$$h = \Delta \tag{8}$$

Using Eq. (8) and Eq. (5), we could have

$$h = \frac{\pi}{\kappa_{\mathrm{c}}} \tag{9}$$

Therefore, for the sharp spectral filter and Gaussian filter, we have

$$h_{\mathrm{ss}} = \frac{\pi}{\kappa_{\mathrm{c,ss}}} = \frac{\pi L}{38} \tag{10}$$

$$h_{\mathrm{G}} = \frac{\pi}{\kappa_{\mathrm{c,G}}} = \frac{\pi L}{54} \tag{11}$$

For 1 Dimensional situation, the ratio of grid numbers is the inverse ratio between grid spacings. And in 3 Dimensional space

$$\frac{N_{\mathrm{G}}}{N_{\mathrm{ss}}} = \left(\frac{h_{\mathrm{ss}}}{h_{\mathrm{G}}}\right)^3 = \left(\frac{54}{38}\right)^3 \approx 2.9 \tag{12}$$

If better resolution are desired with the Gaussian filter and with the filter width unchanged, more grid nodes are needed to reduce the grid spacing, and at the same time, increase κ_{r}. If we want the resolution to be $\kappa_{\mathrm{r}} = 2\kappa_{\mathrm{c}}$, the grid spacing h_{G} has to be decreased to the half of the original grid spacing. This leads to grid number increase by the factor of

$$\left(\frac{h_{\mathrm{ss}}}{\frac{1}{2}h_{\mathrm{G}}}\right)^3 = 2^3 \times \left(\frac{h_{\mathrm{ss}}}{h_{\mathrm{G}}}\right)^3 \approx 23 \tag{13}$$

in 3 Dimensional space.