Solution to Ex. 13.10

of *Turbulent Flows* by Stephen B. Pope, 2000

Yaoyu Hu
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Consider high-Reynolds-number homogeneous isotropic turbulence and an isotropic filter of width Δ and characteristic wavenumber $\kappa_c = \pi/\Delta$. The kinetic energy of the residual motion is

$$\langle k_r \rangle = \int_0^\infty \left[1 - G(\kappa)^2 \right] E(\kappa) \mathrm{d}\kappa \tag{1}$$

For the sharp spectral filter with cutoff κ_c in the inertial subrange, use the Kolmogorov spectrum to obtain the estimate for the fraction of the energy in the residual motions

$$\frac{\langle k_r \rangle}{k} = \frac{3}{2} C (\kappa_c L)^{-\frac{2}{3}} \tag{2}$$

where the lengthscale is $L \equiv k^{3/2} / \varepsilon$. Show that 80% of the energy is resolved (i.e., $\langle k_r \rangle / \mathrm{k} = 0.2$) if κ_c is chosen by

$$\kappa_c L = \left(\frac{15}{2} C \right)^{\frac{3}{2}} \approx 38 \tag{3}$$

Using the relations l_{EI} = $1/6L_{11}$ and L_{11} = $0.43L$ from Section 6.5, show that the corresponding filter width is

$$\frac{\Delta}{l_{EI}} = \frac{6\pi}{0.43 \kappa_c L} \approx 1.16 \tag{4}$$

Perform the same analysis for the Gaussian filter to obtain

$$\frac{\langle k_r \rangle}{k} = C I_0 96^{-\frac{1}{3}} \left(\frac{\Delta}{L} \right)^{\frac{2}{3}} \tag{5}$$

where I_0 is given by

$$I_0 \equiv \int_0^\infty \left(1-e^{-x}\right) x^{-\frac{4}{3}} \mathrm{d}x \approx 4.062 \tag{6}$$

and show that 80% of the energy is resolved for $\kappa_c \equiv \pi / \Delta$ given by

$$\kappa_c L = \pi \left(\frac{5}{96^{\frac{1}{3}}} C I_0 \right)^{\frac{3}{2}} \approx 54 \tag{7}$$

Solution

Using sharp spectral filter and Kolmogorov spectrum Eq. (8), the kinetic energy of the residual motion can be written as Eq. (9)

$$E(\kappa) = C \varepsilon^{\frac{2}{3}} \kappa^{-\frac{5}{3}} \tag{8}$$

$$\begin{aligned} \langle k_r \rangle &= \int_0^\infty \left[1 - G(\kappa)^2 \right] E(\kappa) \mathrm{d}\kappa \\ &= \int_0^\infty \left[1 - H(\kappa_c - \kappa)^2 \right] C \varepsilon^{\frac{2}{3}} \kappa^{-\frac{5}{3}} \mathrm{d}\kappa \\ &= C \varepsilon^{\frac{2}{3}} \int_0^\infty \left[1 - H(\kappa_c - \kappa) \right] \kappa^{-\frac{5}{3}} \mathrm{d}\kappa \\ &= C \varepsilon^{\frac{2}{3}} \int_{\kappa_c}^\infty \kappa^{-\frac{5}{3}} \mathrm{d}\kappa \\ &= \frac{3}{2} C \varepsilon^{\frac{2}{3}} \kappa_c^{-\frac{2}{3}} \end{aligned} \tag{9}$$

The fraction of the energy in the residual motions is

$$\frac{\langle k_r \rangle}{k} = \frac{3}{2} C \frac{\varepsilon^{\frac{2}{3}} \kappa_c^{-\frac{2}{3}}}{k} \tag{10}$$

Use the lengthscale Eq. (11)

$$L = \frac{k^{-\frac{3}{2}}}{\varepsilon} = \left(\frac{\varepsilon^{\frac{2}{3}}}{k} \right)^{-\frac{3}{2}} \tag{11}$$

Eq. (10) turns into

$$\frac{\langle k_{\mathrm{r}}\rangle}{k}=\frac{3}{2}CL^{-\frac{2}{3}}\kappa_{\mathrm{c}}^{-\frac{2}{3}}=\frac{3}{2}C\left(\kappa_{\mathrm{c}}L\right)^{-\frac{2}{3}} \tag{12}$$

Using Eq. (3), Eq. (12) gives

$$\frac{\langle k_{\mathrm{r}}\rangle}{k}=\frac{3}{2}C\left(\kappa_{\mathrm{c}}L\right)^{-\frac{2}{3}}=\frac{3}{2}C\left(\left(\frac{15}{2}C\right)^{\frac{3}{2}}\right)^{-\frac{2}{3}}=\frac{1}{5} \tag{13}$$

The universal Kolmogorov constant, C, is 1.5, this gives

$$\kappa_{\mathrm{c}}L=\left(\frac{15}{2}\times 1.5\right)^{\frac{3}{2}}\approx 38 \tag{14}$$

Based on Eq. (3) or Eq. (14), it is easy to show that

$$\frac{\Delta}{l_{\mathrm{EI}}}=\frac{\frac{\pi}{\kappa_{\mathrm{c}}}}{\frac{1}{6}\times 0.43L}=\frac{6\pi}{0.43\kappa_{\mathrm{c}}L}=\frac{6\pi}{0.43\times 38}\approx 1.154 \tag{15}$$

For the Gaussian filter, Eq. (9) changed into

$$\begin{aligned}\langle k_{\mathrm{r}}\rangle&=\int_0^\infty\left[1-G(\kappa)^2\right]E(\kappa)\mathrm{d}\kappa\\&=\int_0^\infty\left[1-\left(e^{-\frac{\kappa^2\Delta^2}{24}}\right)^2\right]C\varepsilon^{\frac{2}{3}}\kappa^{-\frac{5}{3}}\mathrm{d}\kappa\\&=C\varepsilon^{\frac{2}{3}}\int_0^\infty\left[1-e^{-\frac{\kappa^2\Delta^2}{12}}\right]\kappa^{-\frac{5}{3}}\mathrm{d}\kappa\end{aligned} \tag{16}$$

Let

$$x=\frac{\kappa^2\Delta^2}{12} \tag{17}$$

then

$$\kappa = \left(\frac{12}{\Delta^2} x \right)^{\frac{1}{2}} \tag{18}$$

$$\mathrm{d}\kappa = \frac{6}{\Delta^2} \left(\frac{12}{\Delta^2} x \right)^{-\frac{1}{2}} \mathrm{d}x \tag{19}$$

Substitute Eq. (18) and Eq. (19) into Eq. (16), and considering Eq. (6)

$$\begin{aligned} \langle k_{\mathrm{r}} \rangle &= C\varepsilon^{\frac{2}{3}} \int_0^\infty \left[1 - e^{-\frac{\kappa^2 \Delta^2}{12}} \right] \kappa^{-\frac{5}{3}} \mathrm{d}\kappa \\ &= C\varepsilon^{\frac{2}{3}} \Delta^{\frac{2}{3}} 96^{-\frac{1}{3}} \int_0^\infty \left[1 - e^{-x} \right] x^{-\frac{4}{3}} \mathrm{d}x \\ &= CI_0 96^{-\frac{1}{3}} \varepsilon^{\frac{2}{3}} \Delta^{\frac{2}{3}} \end{aligned} \tag{20}$$

Then the fraction of the energy in the residual motions is

$$\frac{\langle k_{\mathrm{r}} \rangle}{k} = CI_0 96^{-\frac{1}{3}} \Delta^{\frac{2}{3}} \frac{\varepsilon^{\frac{2}{3}}}{k} = CI_0 96^{-\frac{1}{3}} \left(\frac{\Delta}{L} \right)^{\frac{2}{3}} \tag{21}$$

Using Eq. (7), Eq. (21) could be expressed as

$$\begin{aligned} \frac{\langle k_{\mathrm{r}} \rangle}{k} &= CI_0 96^{-\frac{1}{3}} \left(\frac{\Delta}{L} \right)^{\frac{2}{3}} = CI_0 96^{-\frac{1}{3}} \left(\frac{\pi}{\kappa_{\mathrm{c}} L} \right)^{\frac{2}{3}} \\ &= CI_0 96^{-\frac{1}{3}} \left(\frac{\pi}{\pi \left(\frac{5}{96^{\frac{1}{3}}} CI_0 \right)^{\frac{3}{2}}} \right)^{\frac{2}{3}} \\ &= \frac{1}{5} \end{aligned} \tag{22}$$

Therefore, for Gaussian filter, 80% of the energy is resolved for $\kappa_{\mathrm{c}} \equiv \pi / \Delta$.
As for the value of I_0, we apply the method of integration by part on Eq. (6)

$$
\begin{aligned}
I_0 &= \int_0^{\infty}\left(1-e^{-x}\right)x^{-\frac{4}{3}}\mathrm{d}x \\
&= -3\int_0^{\infty}\left(1-e^{-x}\right)\mathrm{d}x^{-\frac{1}{3}} \\
&= -3\left[\left(1-e^{-x}\right)x^{-\frac{1}{3}}\Big|_0^{\infty} - \int_0^{\infty} x^{-\frac{1}{3}}e^{-x}\mathrm{d}x\right] \\
&= \underbrace{-3\left(1-e^{-x}\right)x^{-\frac{1}{3}}\Big|_0^{\infty}}_{*} + \underbrace{3\int_0^{\infty} x^{-\frac{1}{3}}e^{-x}\mathrm{d}x}_{**}
\end{aligned} \tag{23}
$$

The value of the term marked with * in Eq. (23) when x is approaching 0 is evaluated by a limit

$$
-3\lim_{x\to 0}\left(1-e^{-x}\right)x^{-\frac{1}{3}} = -3\lim_{x\to 0}\frac{1-e^{-x}}{x^{\frac{1}{3}}} = -3\lim_{x\to 0}\frac{e^{-x}}{\frac{1}{3}x^{-\frac{2}{3}}} = -9\lim_{x\to 0}e^{-x}x^{\frac{2}{3}} = 0 \tag{24}
$$

And the * term equals zero when x is approaching infinity. Therefore * term is zero. There is only the ** term left, and this term is, in fact, a scaled Gamma function

$$
3\int_0^{\infty} x^{-\frac{1}{3}}e^{-x}\mathrm{d}x = 3\int_0^{\infty} x^{\frac{2}{3}-1}e^{-x}\mathrm{d}x = 3\Gamma\left(\frac{2}{3}\right) \approx 4.0624 \tag{25}
$$

Then, with C = 1.5, we have

$$
\kappa_{\mathrm{c}}L = \pi\left(\frac{5}{96^{\frac{1}{3}}}CI_0\right)^{\frac{3}{2}} \approx \pi\left(\frac{5}{96^{\frac{1}{3}}}\times 1.5\times 4.0624\right)^{\frac{3}{2}} \approx 54 \tag{26}
$$

Finally for Gaussian filter, the corresponding filter width is

$$
\frac{\Delta}{l_{\mathrm{EI}}} = \frac{6\pi}{0.43\kappa_{\mathrm{c}}L} = \frac{6\pi}{0.43\times 54} = 0.81 \tag{27}
$$