Ex. 13.1

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Show from Eq. (13.1) that the operation of filtering and differentiating with respect to time commute, i.e.,

$$\frac{\partial \overline{\mathbf{U}}}{\partial t} = \overline{\left(\frac{\partial \mathbf{U}}{\partial t} \right)} \tag{1}$$

Show that the operations of filtering and taking the mean commute, i.e.,

$$\overline{\left(\langle \mathbf{U} \rangle \right)} = \left\langle \overline{\mathbf{U}} \right\rangle \tag{2}$$

Differentiate Eq. (13.1) with respect to x_j to obtain the result

$$\frac{\partial \overline{U}_i}{\partial x_j} = \overline{\left(\frac{\partial U_i}{\partial x_j} \right)} + \int U_i \left(\mathbf{x} - \mathbf{r}, t \right) \frac{\partial G\left(\mathbf{r}, \mathbf{x} \right)}{\partial x_j} d\mathbf{r} \tag{3}$$

showing that the operations of filtering and differentiation with respect to position do not commute in general, but do so for homogeneous filters.

Solution

Let's differentiate Eq. (13.1) with respect to time

$$\begin{aligned} \frac{\partial}{\partial t} \overline{\mathbf{U}}\left(\mathbf{x}, t \right) &= \frac{\partial}{\partial t} \int G\left(\mathbf{r}, \mathbf{x} \right) \mathbf{U}\left(\mathbf{x} - \mathbf{r}, t \right) d\mathbf{r} \\ &= \int \frac{\partial}{\partial t} G\left(\mathbf{r}, \mathbf{x} \right) \mathbf{U}\left(\mathbf{x} - \mathbf{r}, t \right) d\mathbf{r} \\ &= \int G\left(\mathbf{r}, \mathbf{x} \right) \frac{\partial}{\partial t} \mathbf{U}\left(\mathbf{x} - \mathbf{r}, t \right) d\mathbf{r} \\ &= \overline{\left(\frac{\partial \mathbf{U}}{\partial t} \right)} \end{aligned} \tag{4}$$

$\overline{\left(\langle \mathbf{U} \rangle \right)}$ could be written as

$$
\begin{aligned}
\overline{(\langle \mathbf{U} \rangle)} &= \int G(\mathbf{r},\mathbf{x}) \langle \mathbf{U} \rangle (\mathbf{x}-\mathbf{r},t)\, d\mathbf{r} \\
&= \int G(\mathbf{r},\mathbf{x}) \int_{-\infty}^{+\infty} \mathbf{U}(\mathbf{x}-\mathbf{r},t)\, f(\mathbf{U}) d\mathbf{U} d\mathbf{r} \\
&= \int_{-\infty}^{+\infty} \int G(\mathbf{r},\mathbf{x}) \mathbf{U}(\mathbf{x}-\mathbf{r},t)\, f(\mathbf{U})\, d\mathbf{r} d\mathbf{U} \\
&= \int_{-\infty}^{+\infty} \int G(\mathbf{r},\mathbf{x}) \mathbf{U}(\mathbf{x}-\mathbf{r},t)\, d\mathbf{r} f(\mathbf{U}) d\mathbf{U} \\
&= \int_{-\infty}^{+\infty} \overline{\mathbf{U}} f(\mathbf{U})\, d\mathbf{U} \\
&= \langle \overline{\mathbf{U}} \rangle
\end{aligned} \tag{5}
$$

Eq. (5) is valid since $\overline{\mathbf{U}}$ is function of **U**, (Eq. (3.20)).

Let's differentiate Eq. (13.1) with respect to x_j and obtain

$$
\begin{aligned}
\frac{\partial \overline{U}_i}{\partial x_j} &= \frac{\partial}{\partial x_j} \int G(\mathbf{r},\mathbf{x}) U_i(\mathbf{x}-\mathbf{r},t)\, d\mathbf{r} \\
&= \int \frac{\partial}{\partial x_j} \left(G(\mathbf{r},\mathbf{x}) U_i(\mathbf{x}-\mathbf{r},t) \right) d\mathbf{r} \\
&= \int G(\mathbf{r},\mathbf{x}) \frac{\partial}{\partial x_j} U_i(\mathbf{x}-\mathbf{r},t)\, d\mathbf{r} + \int U_i(\mathbf{x}-\mathbf{r},t) \frac{\partial}{\partial x_j} G(\mathbf{r},\mathbf{x})\, d\mathbf{r} \\
&= \overline{\left(\frac{\partial U_i}{\partial x_j} \right)} + \int U_i(\mathbf{x}-\mathbf{r},t) \frac{\partial}{\partial x_j} G(\mathbf{r},\mathbf{x})\, d\mathbf{r}
\end{aligned} \tag{6}
$$

For homogeneous filter, we have

$$
\frac{\partial}{\partial x_j} G(\mathbf{r},\mathbf{x}) = 0 \tag{7}
$$

Thus Eq. (6) could be written as

$$
\begin{aligned}
\frac{\partial \overline{U}_i}{\partial x_j} &= \overline{\left(\frac{\partial U_i}{\partial x_j} \right)} + \int U_i(\mathbf{x}-\mathbf{r},t) \frac{\partial}{\partial x_j} G(\mathbf{r},\mathbf{x})\, d\mathbf{r} \\
&= \overline{\left(\frac{\partial U_i}{\partial x_j} \right)}
\end{aligned} \tag{8}
$$

This means that for homogeneous filter, the operation of filtering and differentiating with respect to position commute.